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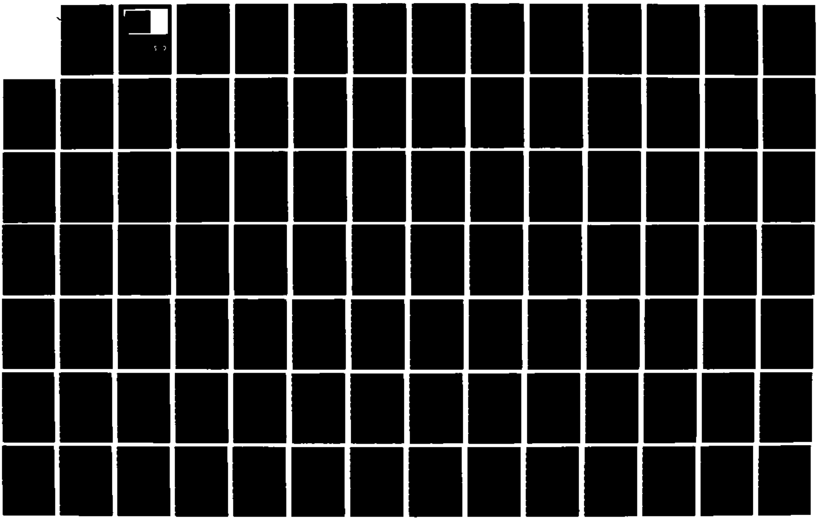

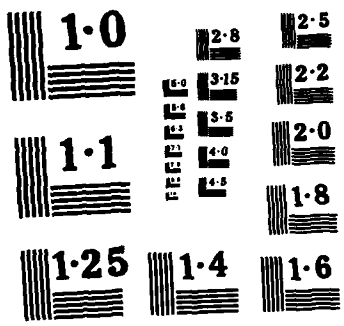

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THE MATHEMATICAL STRUCTURE OF ELEMENTARY PARTICLES - II

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THE MATHEMATICAL STRUCTURE OF ELEMENTARY PARTICLES - II

P. Nowosad*

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ABSTRACT

This report is the second part of a general theory purporting to describe the mathematical structure of the elementary particles, deriving it from first principles. It consists of Chapters ~~VI~~ 6 and ~~VII~~ 7, continuing the MRC Technical Summary Report #2581, October 1983. These chapters ~~In them we~~ study the implications of the sub 2 $SL_2(\mathbb{R})$ transformation group of the particles geometry. In Chapter 6 ~~VI we~~ show how the discrete series of representations implies the quantization of the geometries and in particular why the electron does not interact strongly. In Chapter 7 ~~VII we~~ obtains the resonances as states corresponding to the principal series of representations.

AMS (MOS) Subject Classifications: 81B05, 81G25, 81M05, 83C45, 83E05

Key Words: monochromatic algebras; light quanta; real unimodular group; unitary representations; Poincaré plane; discrete series; principal series; quantization of geometries; resonances; Regge poles

Work Unit Number 2 (Physical Mathematics)

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SIGNIFICANCE AND EXPLANATION

According to Dirac the two fundamental problems that science faces are: the problem of matter and the problem of life. The present work is a contribution toward the analysis of the first problem, in the sense that the true nature of the elementary particles, their fields, masses, interactions, couplings and transformations can be derived from the basic mathematical structures found to be physically relevant, by a step by step analysis of the intrinsic objects pertaining to these structures. Not only are general results obtained but also numerical ones that compare favourably with the experimental data. The relevance of any progress in this direction needs hardly to be stressed.

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THE MATHEMATICAL STRUCTURE OF ELEMENTARY PARTICLES - II

P. Nowosad*

CHAPTER VI. LIGHT QUANTA SYMMETRIES

6.1 INTRODUCTION

In this chapter we consider the effects of the action of the symmetry groups of the geometry on the light waves whose superpositions yield the stationary states (5.69) described in 5.10/12. Henceforth, for convenience, these states will be called harmonic states to witness that they are analytic or anti-analytic in $X^2 + iX^3$ or linear combinations of both.

The main result is that the ratio of the mass of the (bare) particle to that of its basic massive quanta is quantized over the integers, according to the formula

$$m_0/m_0' = \frac{\alpha}{8} \cdot \frac{(n + 1/2)\pi}{1 + |s|} \cdot \frac{\Theta\left(\log \coth \frac{n + 1/2}{1 + |s|} \pi\right)}{\left(\log \coth \frac{n + 1/2}{1 + |s|} \pi\right)^4} . \tag{6.0}$$

Here $n, -s = 0,1,2,\ldots,$ α is the fine structure constant and Θ is the function defined in (5.58), (5.58)'. The integer n stands for the most stable energy level (5.66) of the geometry and s describes the discrete series representation of $SL_2(\mathbb{R})$ in which this occurs: in fact $|s| + 1$ is the winding number of the stable states around the spin-axis.

Condition (6.0) classifies the geometries according to the smallest pair of integers $(1 + |s_0|, n_0)$ that satisfy it. It turns out that for the proton (baryons) $1 + |s_0| = 8$ and $n_0 = 3$, and for the electron (leptons) $1 + |s_0| = 3$ and $n_0 = 0$. These stable energy levels are in perfect agreement with experimental facts, hitherto in lack of theoretical explanation. In particular $n_0 = 0$ explains why the electron does not interact strongly. The dimensions $1 + |s_0| = 8$ and $1 + |s_0| = 3$ turn out to be related to the unitary symmetries of these particles.

The considerations that lead to this result are as follows.

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One observes that the group that preserves the indefinite (r,t) metric in (5.39) contains properly the subgroup that, in addition, preserves the shell of the particle. Whereas the later can be interpreted as a mere change of coordinates of the given system, the action of the quotient subgroup instead gives rise to new, equally valid physical systems, which are however unrelated to each other and so provide no new information. However instead of looking at the action of the quotient subgroup as providing change of coordinates we may alternatively look at it as inducing linear *transformations on a space* of functions on a fiber space where the base is diffeomorphic to the given (r,t) geometry and the fiber is isomorphic to the quotient subgroup. In other words we look for fiber bundle representations of the group action. Then the shell-preserving subgroup yields the so called external symmetries, whereas the fiber representations yield the internal symmetries, i.e. the symmetries of the internal states described by functions on the fiber space.

Distinct (smooth) sections of the fibration represent now the same geometry in distinct states as a result of some kind of excitation imposed on it, which must have a precise physical meaning.

Representations of the quotient subgroup *give rise to induced representations of the* full group on spaces of functions defined on the fiber space (p. 20 [3]). Among these functions the ones representing physically stable states are those that are invariant, up to scalar factors, under the changes of states of excitation of the geometry, that is they are eigenfunctions of the quotient subgroup. The other functions are proper mixtures of these, and as such, unstable.

Distinct inequivalent representations describe distinct gauges of the geometry.

Physical realizability of the particle requires that at least one among the pairs of real monochromatic waves that give rise, by statistical superposition, to stationary states (5.69) describing massive quanta, be (analytic or anti-analytic continuations of) some pair of such stable eigenstates of the fibering subgroup, in some gauge. This singles out the stable energy levels of the quanta of the given particle and the corresponding (stable) representations, and this is how (6.0) is obtained.

Furthermore the physical nature of the excitation of the geometry, as described by the above fibration, is identified to the polarization degree of freedom of the basic monochromatic waves in the twistor field description of the geometry (Section 4.3.8). Thus the above distinct gauges of the geometry correspond to distinct behavior of functions under the polarization group. In this process the quotient subgroup fiber is naturally identified with the unit circle in the complementary (y,z)-space, so that the fiber space has a realization as a subspace of the full geometry. The harmonic stationary states (5.62) are then completely determined as superpositions of the monochromatic waves

$$\psi = (y \pm iz)^{1-s} \cdot e^{i\mu(s-1)(X^4 \pm X^1)} .$$

6.2 THE POINCARÉ HALF PLANE

According to the analysis in Chapter V the metric of an elementary particle is

$$ds^2 = \frac{(dX^1)^2 - (dX^4)^2}{\cosh^2 \mu X^1} + (dX^2)^2 + (dX^3)^2 . \tag{6.1}$$

The r-coordinate in concrete space is given by

$$\tanh \mu X^1 = \pm e^{-\mu|r-r_0|} , \tag{6.2}$$

the sign being the same as the sign of the charge Q of the geometry. As before $X^2 = y$, $X^3 = z$, $X^4 = t$.

Let us restrict ourselves to the two-dimensional metric describing the real isotropic space of the particle, namely

$$d\sigma^2 = \frac{(dX^1)^2 - (dX^4)^2}{\cosh^2 \mu X^1} . \tag{6.3}$$

Since this is a space with constant negative curvature it can be mapped into the Poincaré half-plane, as follows.

Letting γ be a real constant, set

$$\begin{cases} \phi = \xi + \eta = \gamma + e^{\mu(X^4+X^1)} \\ \psi = \xi - \eta = \gamma - e^{\mu(X^4-X^1)} , \end{cases} \tag{6.4}$$

so that

$$(6.5)\qquad \begin{cases} \eta = \dfrac{\phi - \psi}{2} = e^{\mu X^4}\cosh \mu X^1 \\[2ex] \xi = \dfrac{\phi + \psi}{2} = \gamma + e^{\mu X^4}\sinh \mu X^1 , \end{cases}$$

and

$$(6.6)\qquad \begin{cases} d\phi = d\xi + d\eta = \mu e^{\mu(X^4+X^1)}(dX^4 + dX^1) \\[2ex] d\psi = d\xi - d\eta = -\mu e^{\mu(X^4-X^1)}(dX^4 - dX^1) . \end{cases}$$

Consequently

$$(6.7)\qquad d\sigma^2 = \frac{(dX^1)^2 - (dX^4)^2}{\cosh^2 \mu X^1} = \frac{1}{\mu^2}\,\frac{d\xi^2 - d\eta^2}{\eta^2} = \frac{4}{\mu^2}\,\frac{d\phi d\psi}{(\phi - \psi)^2} .$$

Direct calculation shows that the group that preserves (6.7) and the upper half-plane is

$$(6.8)\qquad \phi \longmapsto \tilde{\phi} = \frac{a\phi + c}{b\phi + d}, \qquad \psi \longmapsto \tilde{\psi} = \frac{a\psi + c}{b\psi + d},$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is real and $ad - bc = 1$, i.e. the real 2×2 unimodular group $SL_2(\mathbb{R})$. Since a matrix and its negative yield the same map and both have same determinant the effective group is actually $SL_2(\mathbb{R})/\{1,-1\} \equiv PSL_2(\mathbb{R})$.

In terms of ξ,η

$$(6.9)\qquad \xi \pm \eta \longmapsto \frac{a(\xi \pm \eta) + c}{b(\xi \pm \eta) + d} .$$

If we complexify η into $i\eta$ (6.7) becomes

$$(6.10)\qquad d\sigma^2 = -\frac{1}{\mu^2}\,\frac{d\xi^2 + d\eta^2}{\eta^2} ,$$

which, except for the scalar factor, is the usual Poincaré metric in the upper half-plane $H : \zeta = \xi + i\eta,\ \eta > 0$. As for (6.9), it becomes

$$(6.9)'\qquad \zeta \longmapsto \frac{a\zeta + c}{b\zeta + d} .$$

6.3 GEOMETRICAL CORRESPONDENCE

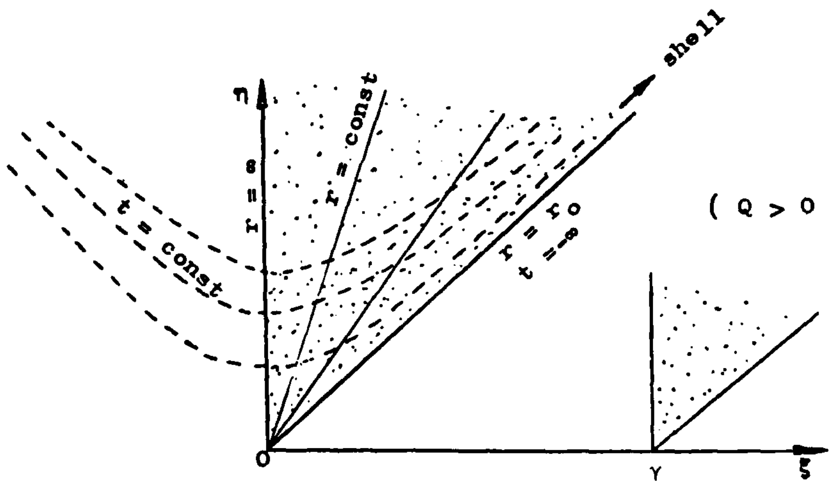
6.3.1 From 6.2 and (6.5) we get

$$\frac{\xi - \gamma}{\eta} = \tanh \mu X^1 = \pm e^{-\mu|r-r_0|}, \tag{6.11}$$

$$\eta^2 - (\xi - \gamma)^2 = e^{2\mu t} > 0 \tag{6.12}$$

Therefore the spheres r = const. correspond to the straight lines $\xi - \gamma$ = const. η through the point $(\gamma, 0)$, with the shell $r = r_0$ corresponding to $\xi - \gamma = \eta$ (for a positively charged geometry) or to $\xi - \gamma = -\eta$ (negatively charged), and infinity $r = \infty$ corresponding to the vertical line $\xi = \gamma$.

Constant times t correspond to the upper branch of the hyperbolae $(\xi - \gamma)^2 - \eta^2$ = const., with remote past $t = -\infty$ also corresponding to $\xi - \gamma = \pm\eta$. In the figure below we consider the case $\gamma = 0$.



In particular the shell of the particle at any time is represented by the point at infinity along the line $\xi - \gamma = \eta$ (or $\xi = \gamma = -\eta$ if $Q < 0$). The finite points on this line are points of the shell only at $t = -\infty$.

Therefore the part of the upper half-plane that corresponds to the physical space is the angular sector A_γ between the line $\xi - \gamma = 0$ and $\xi - \gamma = \eta$ in the upper half-plane, if $Q > 0$, and its reflection A'_γ along the line $\xi = \gamma$ if $Q < 0$. It may be

considered as covered twice, one for the region $x > 0$ and the other for $x < 0$. We have one representation for the region outside the shell and another identical, but independent, for the inside continued into anti-space.

6.3.2 INFINITE-MOMENTUM FRAME

It is clear that η is a time-coordinate and ξ is a space-coordinate. Yet this coordinate system is not physically equivalent to the coordinate system (r,t) describing $d\sigma^2$, in the sense that the elliptic metrics obtained by replacing it for t and $i\eta$ for η do not define the same L^2 spaces of functions. This can be seen directly or by observing that the spatial sections $t = \text{const.}$, which correspond to the hyperbolae $(\xi - \gamma)^2 - \eta^2 = \text{const.}$, are asymptotic to the light-cones along the lines $\xi - \gamma = \pm\eta$ and taking into account Corollary 4, p. 391, and Comments 6, p. 394 in [9].

Furthermore it is also clear that in this referential the particle lies at ∞ and moves with the speed of light (remark physical space outside the particle is the <u>open</u> angular sector A_γ).

The standing point at the vertex, namely $(\gamma,\cdot)$, lying in the left boundary of A_γ corresponds in physical space to a point at infinity $(r = \infty)$. Since the shell of the particle is at rest in the (r,t) referential, in turn it is this point that is moving with the speed of light in physical space, which means the observer with time-axis η is a photon. Thus the (ξ,η) referential describes how an infinitely far photon sees the particle. There are no contradictions in the above statements because we are referring to points in the boundary of our geometry A_γ and the mapping between the two referentials is singular in any neighbourhood of that boundary. In the (ξ,η) referential the particle has infinite momentum. Such referentials are thus called <u>infinite-momentum</u> frames. We will see their role in the electroweak interactions and in parton theory in Chapter X, and how they give rise to renormalization groups.

6.4 GROUP ACTION

The group $SL_2(\mathbb{R})$ may be decomposed into the product KS of the subgroup S of lower triangular matrices $\begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix}$, $a > 0$, and of the subgroup K of rotation matrices

with determinant 1

$$(6.13)\qquad \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad 0 < \theta < 2\pi .$$

This is simply the content of the Gram-Schmidt orthonormalization procedure applied to the columns of any matrix in the group, starting with the second column.

The subgroup S may be decomposed further into the product $\mathcal{D}T$ where T is the subgroup with elements $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ and $\mathcal{D}$ the subgroup $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$, $\alpha > 0$. Clearly $Td = dT$ for every $d \in \mathcal{D}$, because

$$(6.14)\qquad \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ c\alpha^2 & 1 \end{pmatrix} .$$

To fix ideas consider the case when $\gamma = 0$. Then the subgroup that preserves the shell of the particle, i.e. the point at infinity along $\xi = \eta$ (or $\xi = -\eta$ if $Q < 0$), is precisely S because as $\zeta \to \infty$ in (6.9)', $\frac{a\zeta + c}{b\zeta + c} \to \frac{a}{b}$ and as at least one of a and b is non-zero, $\frac{a}{b} = \infty$ if and only if $a \neq 0$ and $b = 0$. In particular $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ takes ζ into $\zeta' = \zeta + c$, i.e. T effects translations parallel to the real axis and $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$ takes ζ into $\alpha^2\zeta$ i.e. $\mathcal{D}$ defines dilatations of the complex plane. In particular this means S preserves the set of rays $r = \text{const.}$ along the vertex of the physical sector $A = A_0$ although it translates the vertex itself.

From (6.5) we see that the maps of $\mathcal{D}$, namely $\xi \longmapsto \alpha^2\xi$, $\eta \longmapsto \alpha^2\eta$ define the <u>time translations</u> $X^4 \longmapsto X^4 + \frac{2}{\mu}\log\alpha$ in physical space. The subgroup T reflects the freedom we have in in choosing the origin along the real axis in (ξ,η) space when defining the metric $\frac{d\xi^2 - d\eta^2}{\eta^2}$, whereas $\mathcal{D}$ does the same with respect to the time origin when defining (6.3).

We may choose to locate the vertex of the physical sector A at any place along the real axis. If its vertex is at $(\xi,\eta) = (\gamma,0)$ we have already written A_γ for it so that in particular $A \equiv A_0$. We then define the action of S modulo the translation that takes the vertex into the point γ, writing $S(\text{mod } T_\gamma)$ for the group so defined. Because of (6.14) we may take T acting on the left or on the right of $\mathcal{D}$, indistinctly, i.e. as

regards $\mathcal{D}$ alone, that is the external symmetries, we might just ignore T altogether. In other words the action of $S(\bmod T_\gamma)$ describes the homotheties with center $(\gamma,0)$.

Due to the importance of the role of the fiber bundle concept in our analysis of the action of K we give an elementary description of the subject of induced representations directly illustrated on the group relevant to us.

6.5 $SL_2(\mathbb{R})$ AS A CIRCLE BUNDLE

6.5.1 The group $SL_2(R)/\{1,-1\}$ is isomorphic to the group of all conformal mappings of the Poincaré half-plane H, as these are the linear fractional transformations of H. Define a linear element as a point of $H \times S^1$, i.e. as a pair formed by a point of H and a direction at that point. Fix a given linear element. Then there is one and just one conformal mapping of H taking any given linear element into the fixed one. Hence the given group may be interpreted as the set of all linear elements in H, or in other words, as a fiber space with base H and fiber S^1 (p. 3 [5]).

It is easy to see that the subgroup that leaves $\zeta_0 = i$ fixed is precisely the orthogonal group K and that an element in S takes i into $\zeta = \xi + i\eta$ if and only if it is of the form

$$\begin{pmatrix} \eta^{1/2} & 0 \\ \xi\eta^{-1/2} & \eta^{-1/2} \end{pmatrix} . \tag{6.15}$$

Since for the matrices in the group, $\frac{d}{d\zeta}\left(\frac{a\zeta + c}{b\zeta + d}\right) = \frac{1}{(b\zeta + d)^2}$, the directions at the original points are rotated by

$$-2 \arg(b\zeta + d) \tag{6.16}$$

at the image points.

Consequently if we decompose the inverse of an element in $SL_2(\mathbb{R})/\{1,-1\}$ in the form

$$\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \pm \begin{pmatrix} \cos\theta/2 & \sin\theta/2 \\ -\sin\theta/2 & \cos\theta/2 \end{pmatrix} \begin{pmatrix} \eta^{1/2} & 0 \\ \xi\eta^{-1/2} & \eta^{-1/2} \end{pmatrix}, \quad \eta > 0, \quad 0 \leq \theta < 2\pi , \tag{6.17}$$

then by (6.9)' this inverse takes i into $\zeta = \xi + i\eta$ and it rotates the directions at

ζ by the amount $-2 \arg(i \sin \theta/2 + \cos \theta/2) = -\theta$, i.e. it takes $(i,0)$ into $(\zeta,-\theta)$. The original element of the group therefore takes $(\zeta,-\theta)$ into $(i,0)$. Hence with ζ and θ given by (6.17) the correspondence

$$\left\{\pm\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right\} \longleftrightarrow (\zeta,-\theta)$$

establishes the above mentioned representation of $SL_2(\mathbb{R})/\{1,-1\}$ as $H \times S^1$, with $(i,0)$ as the fixed linear element.

To define a similar representation for $SL_2(\mathbb{R})$ itself we write

$$(6.18) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \eta^{1/2} & 0 \\ \xi\eta^{-1/2} & \eta^{-1/2} \end{pmatrix}, \quad \eta > 0, \quad 0 \leq \theta < 2\pi .$$

Clearly $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ takes $(\zeta,-2\theta)$ into $(i,0)$. By halving the angles of rotation of the directions we obtain the correspondence $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longleftrightarrow (\zeta,-\theta)$, which yields the representation of $SL_2(\mathbb{R})$ as $H \times S^1$.

The orbits of the points $\zeta = ir$, $0 < r \leq 1$ under the action of K, i.e. the loci

$$\frac{ir \cos\theta + \sin\theta}{-ir \sin\theta + \cos\theta}, \quad 0 \leq \theta < 2\pi ,$$

are circles of radius $\frac{1}{2}\left(r - \frac{1}{r}\right)$ and center $\frac{i}{2}\left(r + \frac{1}{r}\right)$. As they cover H they give all orbits of K. Clearly the rotation of the directions when passing from one to another point of the same orbit is just the rotation of the tangent to the orbit. Thus the image locii L_θ of the segment $\zeta = ir$, $0 < r \leq 1$, under $\begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix}$ are the circles

$$\eta^2 + [\xi + \tfrac{1}{2}(\mathrm{tg}\,\theta/2 - \mathrm{cotg}\,\theta/2)]^2 = 1 + \tfrac{1}{4}(\mathrm{tg}\,\theta/2 - \mathrm{cotg}\,\theta/2)^2$$

orthogonal to the orbits. L_θ are geodesics and together with the orbits of K constitute the net of polar coordinates with center i, in the Poincaré metric.

We take $R = 0.12824$, in which case $\log\coth\frac{R}{2} = 2.748368$. From (6.47) we get

$$|s| + 1 = \frac{\pi}{2.748368}(2n + 1) = 1.143076\ (2n + 1)\ , \tag{6.50}$$

whose values for various n are

$$\begin{array}{ll} n = 0 & |s| + 1 = 1.143076 \\ n = 1 & |s| + 1 = 3.429227 \\ n = 2 & |s| + 1 = 5.715378 \\ n = 3 & |s| + 1 = 8.00153 \end{array} \tag{6.51}$$

This table clearly shows that for the proton the smallest n_0 fulfilling (6.48) (within very reasonable margin of error) is

$$\begin{array}{c} n_0 = 3,\ \text{ with} \\ |s_0| + 1 = 8\ . \end{array} \tag{6.52}$$

From (6.49) we thus see that the set of all solutions for the proton is given by (s',n') with

$$\begin{cases} n' = (2q + 1)3 + q = 7q + 3\ , \\ |s'| + 1 = (2q + 1)8, \quad q = 0,1,2,\ldots\ , \end{cases} \tag{6.53}$$

that is

$$\begin{cases} n' = 3,\quad 10,\quad 17,\quad 24,\ldots \\ |s'| + 1 = 8,\quad 24,\quad 40,\quad 56,\ldots \end{cases} \tag{6.54}$$

Of all the massive quanta associated with the *proton* those with energy level E_3 should be expected to be the most stable. The corresponding representation has index $s_0 = -7$.

6.9.3 CASE OF THE ELECTRON

For the electron the value of R, determined from the experimental data admitted in 5.16, lies in

$$0.74077 < R < 0.74086\ .$$

As seen in Chapter V, R is the basic parameter describing the particle: once R and, say, its mass m_0 are known, we get m_0', r_0, λ, and the coupling constant K.

6.9.1 CONSEQUENCES

The above result shows that the parameter R can only take a denumerable set of values.

The condition (6.45) means that only the lowest weight vector of the corresponding representation can correspond to a monochromatic state.

On the other hand if (s,n) is a solution of (6.48) for a given value of R, then one sees that also

$$((2q+1)s - 2q,\ (2q+1)n + q), \qquad q = 0,1,2,\ldots \tag{6.49}$$

are solutions for the same R.

Therefore along with the stationary state of energy E_n, given by (5.66), and represented by a statistical superposition of the vectors of dominant weight in the s representation by analytic and anti-analytic functions in H, also the states of energy $E_{(2q+1)n+q}$, $q = 1,2,\ldots$, will be stable lowest weight eigenstates of K in the $(2q+1)s - 2q$ representation of the discrete series, respectively, for the same geometry.

From the above analysis we collect the following important consequences.

1. The massive quanta of energy level E_n, where n satisfies (6.48), associated with a given particle, must be more stable than those that do not satisfy it.

2. According to a general rule, the one with smallest energy E_n satisfying (6.48) is expected to be the most stable of all.

We will now apply the results of this analysis to the proton and electron.

6.9.2 CASE OF THE PROTON

For the proton the value of R, determined from experimental data, is given by (5.79):

$$0.12811 < R < 0.12826 .$$

As remarked before, the left-hand sides of (6.43) and (6.44) are complexifications of the monochromatic waves $e^{-im(X^4 \pm X^1)}$, $X^1, X^4 \in \mathbb{R}$ which are given by points of the immersed torus M_γ given in (6.41).

Remark that we cannot achieve the simultaneous realization of the two waves in the same representation: we need K_{-1} for the first one and K_1 for the second.

6.9 QUANTIZATION OF THE GEOMETRIES

As remarked in the Introduction 6.1, to each value of θ in $H \times S^1$ correspond distinct states of the geometry. Functions defined on $H \times S^1$, corresponding to some representation of the discrete series, are stable under the changes of state of the geometry, if and only if they are invariant, up to a scalar factor, under the fibering subgroup K, that is iff they are eigenstates of K in that representation. The other functions are proper mixtures of these and hence unstable.

Since our geometries have been defined in terms of quanta, which are our basic objects, physical realizability of the particle requires that some of the pairs of monochromatic waves that define the quanta (i.e. the stationary states (5.69)), be eigenstates of K_1 or K_{-1} in some representation of the discrete series.

This means that (6.46) must be satisfied for some value of m as given by (5.65), namely

$$m = (n + 1/2)\pi/\rho_0 , \qquad n = 0,1,2,\ldots$$

with $\rho_0 = \frac{1}{2\mu} \log \coth \frac{R}{2}$, $R = \mu r_0$, according to (5.38).

Substituting these values in (6.46) we obtain the condition

$$\log \coth \frac{R}{2} = \frac{2n + 1}{1 + |s|} \pi , \tag{6.47}$$

or

$$R = R(n,s) = \log \coth\left(\frac{n + 1/2}{1 + |s|} \pi\right), \qquad (n,-s = 0,1,2,\ldots) \tag{6.48}$$

Combining (6.48) with (5.77) we get the quantization condition (6.0) for the ratio m_0/m_0'.

We now have the Poincaré half-plane H superposed to our original (ξ,η) plane but we notice that the physical angular sector has its vertex at γ, not necessarily at the origin, and this is reflected in the correspondence (6.42) with the physical coordinates x^1 and x^4. That is the action of the fiber subgroup K on the physical coordinates (and so on the monochromatic waves) is parametrized by the constant γ. We denote this action by K_γ.

6.8 REAL MONOCHROMATIC WAVES AS EIGENSTATES OF K

The real monochromatic waves that satisfied the Laplace-Beltrami equation $\eta_1^2(\partial^2_{\xi_1} - \partial^2_{\eta_1})\phi = 0$ in (ξ_1,η_1)-space, now satisfy

$$\eta_1^2(\partial^2_{\xi_2} + \partial^2_{\eta_1})\phi = 0$$

in H, and so are analytic or anti-analytic in $\zeta = \xi_2 + i\eta_1$.

These functions thus can only correspond to functions in the <u>discrete series of representations of</u> $SL_2(\mathbb{R})$, because the equations (6.28) and (6.30) have non-zero second member $(1 - s^2)\phi$ for the other unitary representations.

From (6.42) we get

$$e^{-m(X^4+X^1)} = \text{const.}\left(\frac{1}{\bar{\zeta} + i\gamma}\right)^{m/\mu}, \tag{6.43}$$

$$e^{-m(X^4-X^1)} = \text{const.}\left(\frac{1}{\zeta + i\gamma}\right)^{m/\mu}. \tag{6.44}$$

Comparing (6.44) with (6.35) and (6.38) with the complex-conjugate of (6.35) namely $\frac{(\bar{\zeta} + i)^k}{(\bar{\zeta} - i)^{k-s+1}}$, we conclude that $e^{-m(X^4+X^1)}$ and $e^{-m(X^4 - X^1)}$ are eigenstates of K_γ if and only if

$$k = 0 \tag{6.45}$$

$$m/\mu = 1 - s \tag{6.46}$$

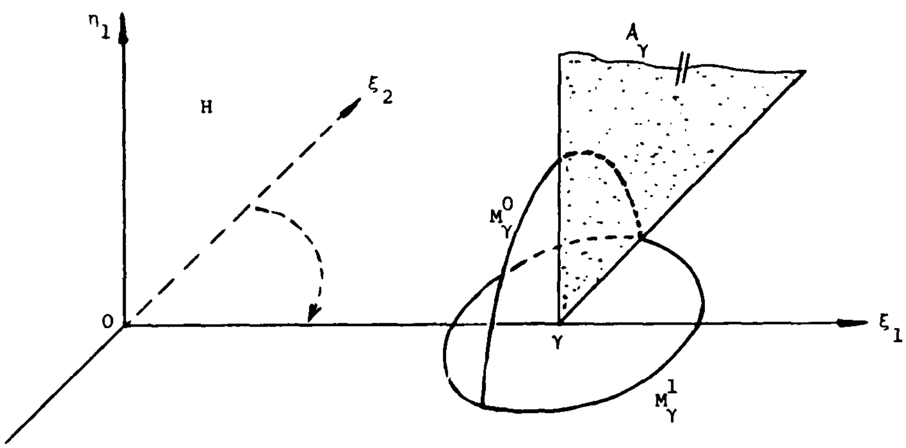
with $\gamma = -1$ for the first wave (the function being anti-analytic in H) and $\gamma = 1$ for the second one (which is analytic).

Notice that M_γ intersects the plane $\eta_2 = 0$ (i.e. the sets where $X^4 = 0$ or $X^1 = \pi/2\mu$) along the two circles

$$M_\gamma^0 : \eta_1 = \cos \mu X^1, \quad \xi_2 = \sin \mu X^1, \quad \xi_1 = \gamma ;$$

$$M_\gamma^1 : \eta_1 = 0, \quad \xi_1 = \gamma - \sin \mu X^4, \quad \xi_2 = \cos \mu X^4 .$$

There are represented in the figure below. Notice that the physical space is the sector A_γ in the (ξ_1,η_1)-plane.



On the plane (ξ_2,η_1) the metric $d\sigma^2$ given by (6.7) is, up to a scalar factor, the definite Poincaré metric.

<u>We identify the half-plane</u> $\eta_1 > 0$ <u>lying there with the half-plane</u> H <u>on which we represent the group</u> $SL_2(\mathbb{R})$ <u>as the fiber space</u> $H \times S^1$. We also rotate it 90° around the η_1-axis to superpose it to the original (ξ_1,η_1)-plane.

Setting $\zeta = \xi_2 + i\eta_1$, (6.39) then becomes in H

(6.42)
$$\begin{cases} \bar{\zeta} + i\gamma = -ie^{\mu(X^4+X^1)} \\ \zeta + i\gamma = ie^{\mu(X^4-X^1)}, \quad \zeta \in H . \end{cases}$$

by i times itself. That means we can also think that we first complexify the coordinates in that local chart and then we identify appropriate n-dimensional sections of it, so that the original and the new metric are defined on the same background.

Following the above lead we complexify all variables ξ, η and X^4, X^1 in (6.4) and (6.7). We then look for some (real) 2-dimensional section of complexified space where the (ξ,η) metric is definite, to identify it with the Poincaré half-plane H and to superpose to our original plane.

Before doing that let us see how the waves stand in the complexification. For that rewrite (6.4) as

$$\text{(6.39)} \qquad \begin{cases} \eta + (\xi - \gamma) = e^{\mu(X^4+X^1)} , \\ \eta - (\xi - \gamma) = e^{\mu(X^4-X^1)} . \end{cases}$$

Replacing in (6.5) iX^1 and iX^4 for X^1 and X^4, respectively, we get in complexified (ξ,η)-space

$$\text{(6.40)} \qquad \begin{cases} \eta = \eta_1 + i\eta_2 = e^{i\mu X^4} \cos \mu X^1 , \\ \xi = \xi_1 + i\xi_2 = \gamma + ie^{i\mu X^4} \sin \mu X^1 , \end{cases}$$

so that

$$M_\gamma : \begin{cases} \eta_1 = \cos \mu X^4 \cos \mu X^1 \\ \eta_2 = \sin \mu X^4 \cos \mu X^1 \\ \xi_1 = \gamma - \sin \mu X^4 \cdot \sin \mu X^1 \\ \xi_2 = \cos \mu X^4 \cdot \sin \mu X^1 , \qquad X^1, X^4 \in \mathbf{R} . \end{cases}$$

Thus when the original variables X^1, X^4 in (6.39) are purely imaginary, the waves in the right-hand side of (6.39) define an immersion M_γ of a torus T^2 in $(\xi_1,\xi_2,\eta_1,\eta_2)$-space. Each point of M_γ represents the pair of values taken by the two waves at their pre-images X^1, X^4 in physical space.

6.7 COMPLEXIFICATION OF THE GEOMETRY

The above unitary representations are realized in the upper half-plane H of $\zeta = \xi + i\eta$ space. Recall that the action of $SL_2(\mathbb{R})$ on H is given by

$$\zeta \longmapsto \frac{a\zeta + c}{b\zeta + d}, \tag{6.38}$$

and that the metric $\frac{d\xi^2 + d\eta^2}{\eta^2}$, and consequently the measure $\frac{d\xi d\eta}{\eta^2}$, are invariant under this group (pp. 41-43 and 181 [7]).

Now we want to relate these representations in the Poincaré half-plane H with the stationary states associated with the particle in the (ξ,η) space with indefinite Poincaré metric (6.7).

As already mentioned, the complexifications $\eta \longmapsto i\eta$ takes the indefinite metric (6.7) into the definite one (6.10). This indicates that complexification is the way to establish these relations. In order to better clarify the meaning of this procedure we first recall the important concept of elliptic metrics associated with a given observer (p. 389 [9]).

Given an n-dimensional pseudo-Riemannian manifold with signature $(1,-1,\ldots,-1)$ say, for short a Lorentz manifold, and given an observer i.e. a unit time-like vector field τ on it, one associates with this pair an elliptic metric (i.e. a proper Riemannian metric) as follows. If the original metric is defined by the bilinear form $g(v,v')$ on vectors of the tangent bundle, then the new (elliptic) metric is defined by

$$g^{\tau}(v,v') = 2g(v,\tau)g(v',\tau) - g(v,v') .$$

This can be visualized as follows. Choose at any given point x a local chart with one coordinate axis tangent to τ at x and with $g_{ij} = \mathrm{diag}(1,-1,\ldots,-1)$ there. Then the matrix of $g(v,\tau)g(v',\tau)$ at x is $\mathrm{diag}(1,0,\ldots,0)$ and so $g^{\tau}_{ij} = \mathrm{diag}(1,\ldots,1)$ at x.

More generally if we take a local chart with Gaussian coordinates $ds^2 = d\tau^2 - \tilde{g}_{ij}dx^i dx^j$ the above procedure yields $(ds^{\tau})^2 = d\tau^2 + \tilde{g}_{ij}dx^i dx^j$.

Alternatively this is the same as replacing the last $n - 1$ coordinates in this local chart by their purely imaginary counterpart or, except for sign, by replacing the first one

The other half of the discrete series is similarly defined for the anti-analytic functions on the upper half-plane $\eta > 0$ (or equivalently analytic functions in the half-plane $\eta < 0$) for each $s = 0,-1,-2,\ldots$.

No two representations of the discrete series are equivalent.

These representations can also be realized by restricting the functions to the real line $\eta = 0$, because by (6.5) this line is given by $\phi = \psi$ and by (6.8) this condition is preserved under the group action.

The only common eigenfunctions of K under these representations are as follows (p. 183, 187 [7]).

For the analytic functions on $\eta > 0$:

(6.35) $$\phi_k(\zeta) = \frac{(\zeta - i)^k}{(\zeta + i)^{k-s+1}}, \qquad k = 0,1,2,\ldots ,$$

with corresponding simple eigenvalue

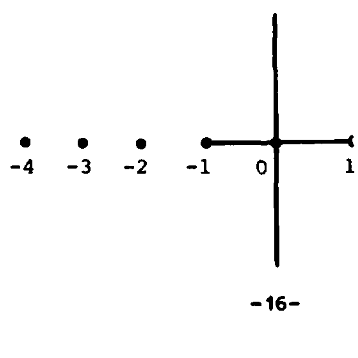
(6.36) $$e^{i(2k+1-s)\theta}, \qquad k = 0,1,2,\ldots$$

Explicitly, if $\gamma = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ then

(6.37) $$T_s(\gamma)\phi_k(\zeta) = e^{i(2k+1-s)\theta}\phi_k(\zeta) .$$

For the anti-analytic functions in $\eta > 0$, just take the complex-conjugate in the above expressions.

These are all the irreducible unitary representations of $SL_2(\mathbb{R})$ (p. 483 [6], p. 45 [2]), besides the trivial one.



s

The above space is realized by an abstract procedure starting from the previous ones and where a central role is played by the unique function in these spaces invariant under K (p. 36 [5]).

In the case of the odd (i.e. $\varepsilon = 1$) principal series a representation may be realized in a certain subspace of functions in the half-plane $\eta > 0$ satisfying the differential equation

$$[-\eta^2(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}) + i\eta(\frac{\partial}{\partial\xi} + i\,\frac{\partial}{\partial\eta})]\phi(\zeta) = \frac{1 - s^2}{4}\,\phi(\zeta)\ , \tag{6.30}$$

where $\zeta = \xi + i\eta$ (p. 57, [5]).

It is given by

$$T_s(\gamma)\phi(\zeta) = \phi(\frac{a\zeta + c}{b\zeta + d})(b\zeta + d)^{-1}\ , \tag{6.31}$$

and commutes with the differential operator in the left hand side of (6.30).

6.6.4 DISCRETE SERIES ($s = 0,-1,-2,\ldots$)

This series has two distinct representations for each value of s: one defined in the upper half-plane $\eta > 0$ and the other in the lower half-plane $\eta < 0$ (p. 36 [5]. p. 480 [6]). They correspond to the representations derived in Section 6.5.

Consider first the case $s \neq 0$. A representation is then defined on the set of analytic functions $\phi(\zeta)$ in the upper half-plane $\eta > 0$, which, together with

$$\hat{\phi}(\zeta) := \zeta^{s-1}\phi(-1/\zeta)\ , \tag{6.32}$$

are of class C^∞ in the closed half-plane $\eta \geq 0$. The representation is given by

$$T_s(\gamma)\phi(\zeta) = \phi(\frac{a\zeta + c}{b\zeta + d})(b\zeta + d)^{s-1}\ , \tag{6.33}$$

with $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and the inner product is

$$\langle\phi_1,\phi_2\rangle = \int_{\eta>0} \phi_1(\zeta)\overline{\phi_2(\zeta)}\eta^{-s-1}d\eta d\xi\ . \tag{6.34}$$

For $s = 0$ the representation is given by (6.33) too and is defined on the set of $L^2(\mathbb{R})$ functions on the real line which are boundary values of analytic functions in $\eta > 0$, equipped with the corresponding inner product.

6.6.1 PRINCIPAL SERIES ($s = i\rho$, $\rho \in \mathbb{R}$, $\varepsilon = 0,1$).

The representations are given on $L_2(\mathbb{R})$ by

$$T_\chi(\gamma)\phi(x) = \phi\left(\frac{ax + c}{bx + d}\right)|bx + d|^{i\rho-1}\operatorname{sgn}^\varepsilon(bx + d) , \tag{6.25}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad - bc = 1$, and χ takes the values above.

Two such representations T_χ, $T_{\chi'}$ are equivalent if and only if $\varepsilon = \varepsilon'$ and $\rho' = \pm\rho$.

6.6.2 SUPPLEMENTARY SERIES ($0 < |s| < 1$, $s \in \mathbb{R}$), $\varepsilon = 0$)

These representations are realized in the space of functions on the real line for which is defined the inner product

$$\langle\phi,\psi\rangle = \int\int |x_1 - x_2|^{-s-1}\phi(x_1)\overline{\psi(x_2)}dx_1dx_2 , \tag{6.26}$$

(for $s > 0$ the integral is in the sense of regularization).

They are defined by

$$T_\chi(\gamma)\phi(x) = \phi\left(\frac{ax + c}{bx + d}\right)|bx + d|^{s-1} , \tag{6.27}$$

with γ as above and $\chi = (s,0)$.

Two representations with indices χ, χ' are equivalent if and only if either $s = s'$ or $s = -s'$.

6.6.3 ALTERNATIVE REPRESENTATIONS OF THE ABOVE SERIES

The even (i.e. $\varepsilon = 0$) principal series and the supplementary series can be also realized on a subspace of the bounded C^∞ functions on the upper half-plane $\eta > 0$, that satisfy the equation (p. 41 [5]):

$$\Delta\phi \equiv -\eta^2\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right)\phi = \frac{1 - s^2}{4}\phi . \tag{6.28}$$

In this case the representation $\chi = (s,0)$ is given by

$$T_s(\gamma)\phi(\zeta) = \phi\left(\frac{a\zeta + c}{b\zeta + d}\right) , \tag{6.29}$$

with $\zeta = \xi + i\eta$, and it commutes with Δ.

functions h the representation T_q is unitary with respect to the above measure because if $\zeta' = \frac{a\zeta + c}{b\zeta + d}$ we have

$$\int_H |T_q h(\zeta)|^2 dA(\zeta) = \int_H |h(\zeta')|^2 dA(\zeta') \; .$$

Observing that

$$\operatorname{Im}\zeta' = \frac{\operatorname{Im}\zeta}{|b\zeta + d|^2} \; , \tag{6.22}$$

one can further define the equivalent representation

$$h(\zeta) \xmapsto{\tilde{T}_q} h\left(\frac{a\zeta + c}{b\zeta + d}\right)\cdot(b\zeta + d)^{-n} \tag{6.23}$$

which is unitary with respect to the new measure $d\tilde{A}(\zeta) = (\operatorname{Im}\zeta)^{n-2} d\zeta d\bar{\zeta} = (\operatorname{Im}\zeta)^n dA(\zeta)$. Indeed, in view of (6.22)

$$\int_H |\tilde{T}_q h(\zeta)|^2 (\operatorname{Im}\zeta)^n dA(\zeta) = \int_H |h(\zeta')|^2 (\operatorname{Im}\zeta')^n dA(\zeta') \; . \tag{6.24}$$

It is immediately clear that the set of analytic functions in the upper half-plane as well as the set of those in the lower half-plane are invariant under (6.23). Further considerations allow the complete identification of the irreducible representations and of the precise spaces of functions involved. At this point, having illustrated how the representations come about, we refer the reader to the literature and we list not just these but all irreducible unitary representations of $SL_2(\mathbb{R})$. As one can see they happen to consist of extensions of (6.23) to appropriate complex exponents.

6.6 IRREDUCIBLE UNITARY REPRESENTATIONS OF $SL_2(\mathbb{R})$

The following are all the irreducible unitary representation of $SL_2(\mathbb{R})$, besides the trivial one. They are indexed by the pair of numbers $\chi = (s,\varepsilon)$ where $s \in \mathbb{C}$, and $\varepsilon = 0$ or 1. (p. 483 [6], p. 33 [5]). (Remark we introduce new notation, distinct from that of previous section).

By restriction to K it is clear that the representation becomes a representation of K; without loss of generality we may assume this is irreducible. In that case the action of K on the fiber variable alone, that is when $f(\zeta,\theta) \longmapsto f(\zeta,\theta+\alpha)$, is necessarily of the form $f(\zeta,\theta) \longmapsto e^{in\alpha}f(\zeta,\theta)$, n a relative integer. Consequently, by (6.19) and (6.20), if $u_\alpha \in K$ we get

$$T_{u_\alpha} f(\zeta,\theta) = f(u_\alpha\zeta,\theta+\alpha) = e^{in\alpha}f(u_\alpha\zeta,\theta) \ .$$

As ζ is arbitrary this implies

$$f(\zeta,\theta+\alpha) = e^{in\alpha}f(\zeta,\theta) \quad \text{i.e.}$$

$$f(\zeta,\alpha) = e^{in\alpha}f(\zeta,0) \ .$$

Hence the sought for subspace of F *consists of functions of the form*

(6.21) $$F_n : f(\zeta,\theta) = e^{in\theta}h(\zeta) \ .$$

In this case it is clear that the action (6.18) yields a representation of $SL_2(\mathbb{R})$, with the action splitting into the canonically induced action in H plus the action of the n-representation of K in S^1, at each $\zeta \in H$.

This representation of $SL_2(\mathbb{R})$ on $H \times S^1$ is called the representation induced by the given representation of its subgroup K.

According to (6.16) the angle α in (6.19), associated with $q = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ at the point ζ, is given by

$$\alpha = -\arg(b\zeta + d)$$

so that

$$e^{i\alpha} = \left(\frac{b\zeta + d}{|b\zeta + d|}\right)^{-1} \ .$$

Therefore

$$T_q f(\zeta,\theta) = e^{in\alpha}f(q\zeta,\theta) = f(q\zeta,\theta)\left(\frac{b\zeta + d}{|b\zeta + d|}\right)^{-n}$$

In particular, setting θ = const., we get the representation on functions on H alone

$$h(\zeta) \xrightarrow{T} h\left(\frac{a\zeta + c}{b\zeta + d}\right)\left(\frac{b\zeta + d}{|b\zeta + d|}\right)^{-n} \ .$$

The element of area in H, associated with the Poincaré metric in (6.10), is clearly $dA(\zeta) = \frac{d\xi d\eta}{\eta^2} = \frac{d\zeta d\bar{\zeta}}{(\mathrm{Im}\zeta)^2}$, and it is invariant under $SL_2(\mathbb{R})$. In an appropriate subspace of

On the other hand, according to (6.16) the elements of S do not change directions of the line elements, so they do not act on referentials of the fiber. This is also clear from the fact that the orbits of T are the lines parallel to the real axis and those of $\mathcal{D}$ are the straight lines through the origin (so tangent directions are unchanged).

As a consequence one can immediately determine the matrix in the group, factored as KS, that maps a given point $\zeta_1 \in L_{\theta_1}$ and its corresponding fiber referential into another point $\zeta_2 \in L_{\theta_2}$ and its associated fiber referential: find the intersection ζ_0 of the K-orbit of ζ_1 with L_{θ_2}, apply u_θ that takes ζ_1 into ζ_0 and then the matrix $r \in S$ that takes ζ_0 into ζ_2. This applies also when one or both points coincide with i, depending on which fiber referentials are there associated with these points; in particular when one point, say, $\zeta_2 \neq i$, we consistently associate with $\zeta_1 = i$ the same fiber referential of ζ_2, for the above purpose.

It is clear therefore from the above analysis, that the action of $q = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ on $H \times S^1$, induced from its action on $SL_2(\mathbb{R})$ itself, can be described by

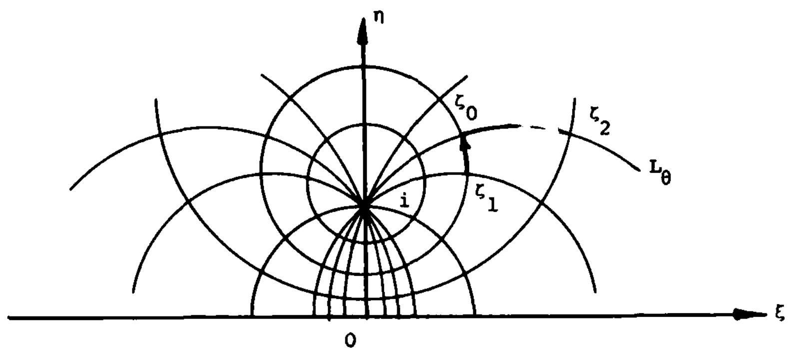
$$(6.19) \qquad \begin{cases} q(\zeta,\theta) = (q\zeta, \theta + \alpha)\ , \quad \text{where} \\[1em] \quad q\zeta := \dfrac{a\zeta + c}{b\zeta + d}\ , \quad \text{and} \\[1em] \quad \alpha = \dfrac{\theta_2 - \theta_1}{2}\ , \quad \text{where} \quad \zeta \in L_{\theta_1},\ \ q\zeta \in L_{\theta_2}\ . \end{cases}$$

6.5.2 REPRESENTATIONS ON $H \times S^1$

There is a standard process, due to Frobenius, to construct a linear representation of a group starting from linear representations of certain of its subgroups (see p. 16 [3], p. 17 [2]). This can be done in a general setup, but we may do it using our previous analysis.

Consider the space F of complex-valued functions on $H \times S^1$, and look for a representation of $SL_2(\mathbb{R})$ on a subspace of F, given by the canonical action on $H \times S^1$, namely $q \longmapsto T_q$, where

$$(6.20) \qquad T_q f(\zeta,\theta) = f(q(\zeta,\theta))\ .$$



θ = 0

The orbits and their orthogonal trajectories thus describe the action of any element of K on $H \times S^1$, as induced from their action on $SL_2(\mathbb{R})$.

Indeed, given two points on the same orbit of K, lying in L_{θ_1} and L_{θ_2} say, the previous analysis shows that the first is taken into the second one by $u_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, with $\theta = \dfrac{\theta_2 - \theta_1}{2}$. To complete the description we represent the fiber S^1 at each point distinct from i, as a circle normal to and touching the H-plane from above, and choose the origin of the angles so that, at any point of L_θ, it determines the angle $-\theta$ with respect to the tangent point. At the point i, where all L_θ, meet, we have the usual angle indeterminacy. For convenience we replace $\{i\} \times S^1$ by the family $\{i\} \times S^1_\theta$, $0 \leq \theta < 2\pi$, where S^1_θ stands for S^1 with the referential it has in L_θ.

This choice of referentials on the fibers automatically incorporates a geometric representation of the action of K on $H \times S^1$: at the same time as K moves a point along an orbit, the original angles are increased by the corresponding amount $\Delta\theta$, because the origin of the referential is displaced by $-\Delta\theta$ [we are thus representing the action of K on S^1 as a change of referential, not as a map of S^1, whose points are thus kept fixed]. At the point i the effect of K is to change from one to another S^1_θ, while i remains fixed.

We take $R = 0.74080$ so that $\log \coth \frac{R}{2} = 1.03749$. From (6.48) we get

$$|s| + 1 = \frac{\pi}{1.03749}(2n + 1) = 3.02806\,(2n + 1)\,, \tag{6.55}$$

which yields for $n = 0$ the value

$$|s| + 1 = 3.02806\,.$$

This value is sufficiently close to 3 that the error, which is less than 1%, may be ascribed both to the highly indirect experimental method of determination of the mass of the muon (p. 203, [8]) as to the fact that we computed the mass of the pair neutrino anti-neutrino $\nu + \bar{\nu}$ by a simple difference in Section 5.16 and assumed it to play the role of the electron quanta.

Therefore for the electron the most stable states correspond to $n_0 = 0$ in the $s_0 = -2$ representation (i.e. $|s_0| + 1 = 3$) of the discrete series.

From (6.49) also

$$n' = q, \quad s' = -(6q + 2), \quad q = 0,1,2,\ldots \tag{6.56}$$

yield allowed states. In particular $|s'| + 1 = 3(2q + 1)$.

The corresponding numerical values are

$$\left\{\begin{array}{rcccc} n' = 0, & 1, & 2, & 3,\ldots \\ |s'| + 1 = 3, & 9, & 15, & 21,\ldots\,. \end{array}\right. \tag{6.57}$$

Notice $|s'| + 1$ are the odd multiples of 3.

6.9.4 COMPARISON WITH EXPERIMENTAL FACTS

By the previous analysis the most stable massive quanta for the proton should have energy level E_3, which corresponds to a system of three pions $3\pi°$.

Similarly the most stable quanta energy level for the electron is E_0, which means no quanta at all.

If we consult a table of mesons (p. 402, [1]) we get the following data

quanta	decay mode	mean life	energy level
$\pi°$	$2\gamma(100\%)$	1.8×10^{-16}sec	$n = 1$
K_S°	$2\pi°(31\%)$	0.8×10^{-10}sec	$n = 2$
K_L°	$3\pi°(34\%)$	5.2×10^{-8} sec	$n = 3$

As we see, the kaon L, which decays into $3\pi°$, is about 600 times more stable than the kaon S, which decays into $2\pi°$, and 3×10^8 times more stable than the $\pi°$.

Furthermore the meson resonance η, which decays into $\pi^+\pi^-\pi°(25\%)$ or $3\pi°(30,8\%)$ (p. 3, [4]) is an almost stable particle, as far as strong interactions are concerned (p. 81, [4]).

These experimental facts therefore are in agreement with the theoretical analysis, which thus explains this exceptional behavior.

As for the electron it is well known that they do not interact strongly i.e. by the exchange of massive quanta (they only interact electromagnetically and weakly). This is also predicted by our symmetry analysis above, which tells us that its most stable stationary state is the state with no quanta, i.e. with energy E_0.

6.9.5 PRECISE VALUE OF THE RATIO m_0/m_0'

We have identified the proton as a $(s_0,n_0) = (-7,3)$ particle, and the electron as a $(-2,0)$ particle.

The precise R values are then, by (6.48),

(6.58) $$R_p = \log \coth \frac{7\pi}{16} = 0.128172 ,$$

(6.59) $$R_e = \log \coth \frac{\pi}{6} = 0.732985 ,$$

where the suffix refers to the particle.

These are the values that we used as the end of previous chapter, and the reason for it is now clear.

We then compute $\Theta(R)$, where Θ is defined by (5.58)', so that

$$\Theta(R_p) = 1.4990 \quad \text{and} \quad \Theta(R_e) = 3.0647 .$$

From (5.77) and (6.48) we have

$$\frac{m_0'}{m_0} = \frac{16}{\alpha} \cdot \frac{R^4}{\Theta(R)} \cdot \frac{|s| + 1}{(2n + 1)\pi} , \tag{6.60}$$

so that for the proton $(n_0 = 3, \ s_0 = -7)$

$$\frac{m_0'}{m_0} = 16 \times 137.036 \times \frac{2.69881 \times 10^{-4}}{1.4990} \times \frac{8}{7\pi} = 0.143604$$

i.e.

$$\frac{m_0}{m_0'} = 6.9636 . \tag{6.61}$$

The experimental value is 6.9514, hence the relative error is 0.18%.

For the electron $(n_0 = 0, \ s_0 = -2)$

$$\frac{m_0'}{m_0} = 16 \times 137.036 \times \frac{0.288656}{3.0647} \times \frac{3}{\pi} = 197.205 , \tag{6.62}$$

whereas the value that we used in Section 5.16 was 205.77, a difference of 4.6%. (We shall actually see in Chapter VII, that the muon is more likely a resonance state, not an internally excited state of the electron).

6.9.6 COMMENT

The numerical agreement (excellent for the proton, reasonable for the electron) of the theoretical with the experimental ratios m_0'/m_0, over such a wide range of values (0.1436 to 197.205), certainly rules out any coincidence due to pure chance. Furthermore the theoretical anticipation of the higher stability of the E_3 energy level for the proton, i.e. for a system of three pions, and of the E_0 level for the electron, i.e. for no massive electron quanta at all, confirms the relevance of the above group-theoretical analysis of the geometries and explains fundamental facts regarding the exchange of massive quanta, that is, the strong interactions.

6.10 POLARIZATION AND THE SUBGROUP K

By the previous analysis the appropriate complex continuations of the real monochromatic states $e^{i\mu(s-1)(X^4 \pm X^1)}$ are eigenstates of $K_{\pm 1}$ with eigenvalues $e^{\pm i(s-1)\theta}$, respectively.

The corresponding functions on $H \times S^1$ defined by (6.21) are thus, respectively,

$$e^{i\mu(s-1)[X^4 \pm (X^1 + \theta/\mu)]} . \tag{6.63}$$

On the other hand, in our analysis of the harmonic stationary states in 5.10.1, the coefficients A,B in (5.62) were analytic or anti-analytic functions of $y + iz$, which remained indeterminate. To fix ideas consider in particular the real wave $e^{i\mu(s-1)(X^4 - X^1)}$, with eigenvalue $e^{i(1-s)\theta}$. If we take as its coefficient in (5.62) the analytic function of $y + iz$, $A = (y + iz)^{1-s}$ we obtain

$$\psi = (y + iz)^{1-s} \cdot e^{i\mu(s-1)(X^4 - X^1)} . \tag{6.64}$$

Consider now the twistor representation of the geometry as described in 4.3.7. As shown in 4.3.8 if the polarizations of the spinor fields that describe the two real monochromatic waves are changed by amounts that differ by the angle $\tilde{\theta}$, then the plane $y + iz$ is rotated by this same angle $\tilde{\theta}$. In this case (6.64) goes into

$$\psi \longmapsto \psi e^{i(s-1)\tilde{\theta}} . \tag{6.65}$$

Had we taken instead $A = (y - iz)^{1-s}$, the transformation would have been

$$\psi \longmapsto \psi e^{-i(s-1)\tilde{\theta}} . \tag{6.66}$$

Consequently the action of K on the eigenstates (6.63) in the representation of the discrete series with index s, is precisely the same action as that produced on the harmonic states (6.65) and (6.66) due to the *change* of relative polarization of the *incoming* and *outgoing* monochromatic waves of the particle. Furthermore (6.63) can be lifted into (6.64) by means of the factor $(y^2 + z^2)^{\frac{1-s}{2}}$.

Therefore we can identify the fiber S^1, on which K acts, with the unit circle on the *complex* isotropic plane $y + iz$, on which the relative polarization group of the

incoming and outgoing waves acts, and so also equate their effects, provided we take for coefficients of the real monochromatic waves powers $1 + |s|$ of $y + iz$ or $y - iz$.

This eliminates the indeterminacy in the waves that constitute the stationary states (5.62) and gives the gauge theory associated with the representation of $SL_2(\mathbb{R})$ induced by its subgroup K, a concrete physical meaning. It even provides a realization of $SL_2(\mathbb{R})$ in terms of the full geometry.

The distinct possible polarization gauges of the geometry are thus indexed by the different circulation, or winding numbers, of the vector fields given by the waves (6.64), around the spin-axis $y = z = 0$, computed for a fixed time. The only stable gauges for a given geometry are those with winding number the odd multiples of the minimum solution $1 + |s|$ of the quantization condition (6.0) for that geometry. In particular for the proton (baryons) these are 8, 24, 40, 56, 72,... and for the electron 3, 9, 15, 21, 27,... .

We will see in Chapter VIII that the nodal lines $y + iz = 0$ of ψ do not carry additional energy, besides the time-vibrational energy of the real waves.

The multiplicative superposition of the real and of the complex waves in (6.64), imposed by the action of the relative polarization group Π, confirms that the massive quanta of the geometry are indeed the statistical superposition of photons, in accordance with their description in 2.4.1. The additive superposition as in (2.3.1), which is also allowed, leads to neutrino fields, which however do not obey the conditions imposed by the polarization group symmetry; that is, they do not belong in the discrete representations.

As a final comment the quantization condition (6.0), as expressed in (6.63), simply says that, if the variables x^4, x^1 and θ are corrected according to the relative curvature of the unit circle in the space where these coordinates are immersed (leading thus to x^4, x^1 and θ/μ) then the stable harmonic states are those having equal frequencies in all the corrected variables. Thus, this just becomes an instance of the familiar Bohr-type rule of quantization.

The complete harmonic stationary states (5.69) are given by

$$\phi = e^{i\mu(1 - s)X^4} \cdot (X^2 \pm iX^3)^{1-s} \sin \mu(1 - s)X^1 \ , \tag{6.67}$$

or making explicit the energy level, by

$$\phi = e^{i\left(n+ \frac{1}{2}\right)\frac{\pi}{\rho_0} X^4} \cdot (X^2 \pm iX^3)^{1-s} \sin\left(n+ \frac{1}{2}\right) \frac{\pi}{\rho_0} X^1 \ . \tag{6.68}$$

For the proton $n_0 = 3$, $1 - s_0 = 8$ so that the stable stationary states are

$$\phi_p = e^{i\left(3+ \frac{1}{2}\right)\frac{\pi}{\rho_0} X^4} (X^2 \pm iX^3)^8 \sin\left(3 + \frac{1}{2}\right) \frac{\pi}{\rho_0} X^1 \tag{6.69}$$

and for the electron $n_0 = 0$, $1 - s_0 = 3$ so that

$$\phi_e = e^{i \frac{\pi}{2\rho_0} X^4} (X^2 \pm iX^3)^3 \sin \frac{\pi}{2\rho_0} X^1 \ . \tag{6.70}$$

Whereas the stable fields (6.70) for the electron are non-constant, they cannot be detected by the emission of massive quanta because $n_0 = 0$ for them. One can only expect to detect their existence by means of experiments outside the field of strong interactions. They are thus apparently related to the Higgs fields of electroweak interactions.

6.11 UNITARY SYMMETRY OF THE STABLE QUANTA

We still need to analyze the implications of the fact that two distinct representations, K_1 and K_{-1}, are needed to describe the outgoing and incoming waves that originate the quanta of the geometry.

Recall that $e^{-im(X^4+X^1)}$ is an outgoing wave for a positively charged geometry and an incoming wave for a negative one, while $e^{-im(X^4-X^1)}$ is incoming when $Q > 0$ and outgoing when $Q < 0$ (cf. (5.70)).

Therefore when $m/\mu = 1 - s$, in the K_1 representation the analytic eigenstate $\frac{1}{(\zeta + i)^{1-s}}$ stands for an incoming wave when $Q > 0$ and an outgoing one when $Q < 0$, and the eigenvalue is $e^{i(s-1)\theta}$. Analogously in the K_{-1} representation the anti-analytic eigenstate $\frac{1}{(\bar{\zeta} - i)^{1-s}}$ stands for an outgoing wave when $Q > 0$ and an incoming one when $Q < 0$, and the corresponding eigenvalue is $e^{-i(s-1)\theta}$. More specifically we have the states

K_1 :

$$e^{-m(X^4 - q|X^1|)} = \text{const} \frac{1}{(\zeta + i)^{1-s}}, \quad (X^2 + iX^3)^{1-s}$$

(analytic)

K_{-1} :

$$e^{-m(X^4 + q|X^1|)} = \text{const} \frac{1}{(\bar{\zeta} - i)^{1-s}}, \quad (X^2 - iX^3)^{1-s}$$

(anti-analytic)

Here $q = \text{sgn}\, Q$, with Q = charge of the geometry.

The factors in the extreme right side express the appropriate behavior under the polarization group. In fact the number $\gamma = \pm 1$ in the K_γ representations is thus seen to index the polarization orientation of the photon waves (6.64) and (6.65) for we may say that (6.65) and (6.66) have oppositely oriented polarizations (for short, opposite polarizations).

In order to produce the $\sin \mu(s - 1)X^1$ factor in the stationary states (6.67) we must add photon waves with same polarization, that is, we must superpose eigenstates in the same representation K_γ. But then the only alternative is to take subjacent geometries with opposite charge. In this way we can obtain the stationary harmonic states

$$e^{i\mu(s - 1)X^4} \cdot (X^2 \pm iX^3)^{1-s} \cdot \sin \mu(1 - s)X^1 , \tag{6.71}$$

with $\pm$ sign according as $\gamma = \pm 1$. For short we may condense the expression for the photon wave corresponding to charge sign q and polarization orientation γ in the form

$$e^{-m(X^4 - \gamma q|X^1|)} \cdot (X^2 + i\gamma X^3)^{1-s} , \tag{6.72}$$

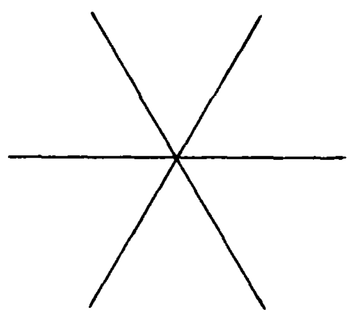
and call it the K_γ^q -representation with $q,\gamma = \pm 1$, where now we have also explicited the choice of q in the representation. For convenience we write K_γ^q itself to represent the above photon wave. In that case the stationary states (6.71) are given by

(6.73) $$K_\gamma^1 + K_\gamma^{-1}, \qquad \gamma = \pm 1 .$$

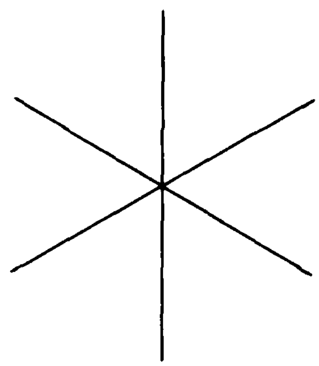
However we may have new states composed of the statistical superposition of photons with opposite polarization. Thus K_1^1 and K_{-1}^{-1} are both incoming photon waves and their superposition $aK_1^1 + bK_{-1}^{-1}$ has the form, (in the particular case that $1 - s = 3$ (electron)):

$$e^{-m(X^4-|X^1|)}\{(a + b)y(y^2 - 3z^2) + i(a - b)z(3y^2 - z^2)\} .$$

Here we have two special cases: when $a + b = 0$ or $a - b = 0$, $a^2 + b^2 \neq 0$. In the first case the nodal set in (y,z) space is given by $z = 0$ and $z = \pm\sqrt{3}\, y$, and in the second case, by $y = 0$ and $y = \pm\sqrt{3}\, z$. On is obtained from the other by a 90° rotation.



a = b



a = -b

In all other cases the nodal set is the intersection of these two sets, which is just the origin $y = z = 0$. It is thus seen that only in the two particular cases above the nodal set of the superposition of the two incoming oppositely polarized photon waves is diffused on three planes instead of being localized on a line $(y = z = 0)$ in 3-space. Of course these two states can be used as a basis of the space $aK_1^1 + bK_{-1}^{-1}$.

In the general case $1 - s$, $-s = 0,1,2,\ldots$ we will have again the same two diffused states as above, except that their nodal sets are the $1 + |s|$ planes where $\cos(1 - s)\theta = 0$ and $\sin(1 - s)\theta = 0$, respectively.

If we superpose diffused outgoing and incoming photon waves we can now get stationary diffused harmonic states.

This procedure means we are additively superposing monochromatic waves of the algebras N, N', and $\tilde{N}$, $\tilde{N}'$ defined in 5.10, (which thus remain harmonic) and so give the most general stable harmonic states solutions we were looking for.

At any rate we have a 4-dimensional space generated by the photon waves K_γ^q, $\gamma, q = \pm 1$ expressing their possible quantum-mechanical superpositions. We can take these four states as mutually orthogonal in a natural way because $(X^2 + iX^3)^{1-s}$ and $(X^2 - iX^3)^{1-s}$ are orthogonal on S^1 with respect to the Lebesgue measure and so are the outgoing and incoming waves in the Poincaré plane with respect to the invariant measure there.

In particular this means that the symmetry group of the stable photon waves is SU_4 and that it contains as subgroup the charge-conjugation permutation subgroup C and the polarization-orientation subgroup Π, both subgroups being mixed in SU_4, as shown in particular in (6.73). These subgroups act originally on the real monochromatic waves or in the spinor representations of these.

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CHAPTER VII. RESONANCES

7.1 INTRODUCTION

In this chapter we determine which non-harmonic time-periodic solutions of the wave equation $\Delta_2\varphi = 0$ belong, after analytic continuation, in some (possibly extended) unitary representation of $SL_2(\mathbf{R})$.

The main result is that the free states called resonances belong in a denumerable subset of the principal series representations, with their total mass m_N being given by the following general expression in terms of the bare mass m_o

$$\frac{m_N}{m_o} = 1 + \frac{8}{\alpha} \cdot \frac{1+|s_o|}{(n_o+1/2)\pi} \cdot \frac{\left(\log \coth \frac{n_o+1/2}{1+|s_o|}\pi\right)^4}{\Theta\left(\log \coth \frac{n_o+1/2}{1+|s_o|}\pi\right)} \cdot \tag{7.0}$$
$$\cdot\left\{n' + \left[1 + 4\left(\frac{n_o+1/2}{1+|s_o|}\right)^2 \Lambda_N\right]^{1/2}\right\}, \quad N = 1,2,\ldots .$$

Here $(1+|s_o|, n_o)$ are the quantization integers for the geometry, α is the fine structure constant, Θ is the function defined in (5.58), $n' = \max(0,n_o-1)$ if the balance of energy allows it, otherwise $0 \leq n' \leq \max(0,n_o-1)$, and $\sqrt{\Lambda_N} =$
$= \frac{1}{4}\sqrt{4N(N+1)-1} + c_N$ with $c_N \cong 0.10$ for $N \geq 2$ and $c_1 \cong 0.09$.

In the case of the proton, namely when $n_o = 3$, $1+|s_o| = 8$, we get for $n' = 1$ and $N = 1$ the mass $E_1 = 1{,}234.7$ MeV (in electrostatic units), which is close to the mass of the first baryon resonance, namely 1236 MeV, usually denoted by $N^*(1238)$. For $N > 2$, $n' = n_o-1 = 2$ is possible and thus takes place.

For $N = 3$ and $N = 4$ we then get $E_3 = 1461$ MeV, $E_4 = 1514$ MeV which agree with the masses of $N^*(1460)$ and $N^*(1515)$, respectively. And so on.

For the electron $n_o = 0$, $1+|s_o| = 3$ and $n' = 0$. We then get

$$
\begin{aligned}
N &= 1, & E_1 &= 104.38 \text{ MeV} \\
N &= 5, & E_5 &= 138.54 \text{ MeV} \\
N &= 28, & E_{28} &= 492.82 \text{ MeV} \\
N &= 52, & E_{52} &= 891.27 \text{ MeV} \\
N &= 56, & E_{56} &= 958.06 \text{ MeV}
\end{aligned}
$$

which compare well with the masses (in MeV)

$$
E_{\mu^\pm} = 105.66 \qquad E_{\pi^\pm} = 139.58, \qquad E_{K^\pm} = 493.78,
$$

$$
E_{K^{*\pm}} = 891\pm1, \qquad E_{\eta'} = 959\pm2, \quad \text{respectively}
$$

The decays $\mu^- \to e\nu\bar{\nu}$, $\pi^- \to \mu^-\nu$, $K^- \to \mu^-\nu$ further confirm that, from the dynamical point of view, such particles are indeed resonances of the electron, and that $\nu\bar{\nu}$ substitutes for the (photonic) massive quanta of the electron, found to be unstable in Chp. VI.

The wave-equation solutions defining the resonances are given by

$$
e^{\pm i2\mu\sqrt{\Lambda_N}t} \cdot P^{\pm 2i\sqrt{\Lambda_N}}_{-\frac{1}{2}+i\frac{\rho_N}{2}}\left(e^{-\mu|r-r_o|}\right) \cdot J_o\left[\frac{\mu}{2}\left(1+\rho_N^2\right)^{1/2}\left(y^2+z^2\right)^{1/2}\right],
$$

with $\rho_N = \sqrt{4N(N+1)-1}$, $N = 1,2,\ldots$

The symmetry group describing their quantum-mechanical superpositions is SU_4, and time-inversion T and charge-conjugation C are <u>mixed</u> in it, when combined to form the solutions

that are standing solutions at $r = r_o$.

These results are obtained as follows.

Separation of variables leads to the eigenvalue equation $\eta^2(\partial_\xi^2 - \partial_\eta^2)U = \frac{1-s^2}{4}U$ in the Poincaré half-plane, and to the Schrödinger operator on the real line $-\frac{d^2}{dx^2} + \frac{1-s^2}{4\cosh^2\frac{x}{2}}$.

Time-periodic non-harmonic solutions which are C^∞ and bounded, when continued into the (elliptic) Poincaré half-plane, exist if and only if $s \in (-1,1) \cup i\mathbb{R}$. This, together with the eigenequation above, imply that they correspond to states in the supplementary or in the even principal series. The spectrum of the Schrödinger operator is then purely continuous and consists of the positive semi-axis $\Lambda \geq 0$. Thus these solutions represent free states, i.e. they describe scattering phenomena associated with the given geometry.

Analytic continuation of the metric (5.0) into the region $r \in (-\infty, r_o)$, once we replace $(r-r_o)$ for $|r-r_o|$, yields a second (singular) solution with $\mathrm{cosech}^2\frac{x}{2}$ in place of $\mathrm{sech}^2\frac{x}{2}$. These solutions, associated with analytic continuation of the regular solutions, seem to be characteristic to resonance phenomena and may be called shadow solutions (see for example p.47 [11]). In fact we shall show in Chp. X that their role is to restore conformal flatness at infinity, by statistical superposition with the regular metric, and so they may be considered as resulting from a boundary condition at infinity.

For s in the above range of values, this singular potential is <u>reflectionless</u> (p.435[4]), (i.e. the shell of the particle is transparent to incident waves) if and only if $1-s^2 =$

$= N(N+1)$, $N = 1,2,\ldots$ and this implies

$$s = i\rho = i\sqrt{4N(N+1)-1}, \qquad N = 1,2,\ldots$$

That is to say, under the above conditions, only the principal series is allowed and then just for the discrete set of values $s_N = i\rho_N$ above. This elimination of the supplementary series adds to the fact that, of both representations, only the principal one is induced by the representation of a subgroup, namely the subgroup $\mathcal{D}$ of diagonal matrices $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$, which describes the time-translations. Thus the allowed scattering states are eigenstates of time, considered as the parameter of the fiber in the induced representation rather than as a coordinate in space-time.

The Schrödinger operator above has the spectral density

$$\mathrm{Im}\,\frac{\Gamma[\frac{1}{4} - i(\frac{\rho}{4} + \sqrt{\Lambda})]\Gamma[\frac{1}{4} + i(\frac{\rho}{4} - \sqrt{\Lambda})]}{\Gamma[\frac{3}{4} - i(\frac{\rho}{4} + \sqrt{\Lambda})]\Gamma[\frac{3}{4} + i(\frac{\rho}{4} - \sqrt{\Lambda})]},$$

and this has a unique point of absolute maximum at $\sqrt{\Lambda_1} \cong 0.75$ for $N = 1$, and at

$$\sqrt{\Lambda_N} \cong \frac{\rho_N}{4} + 0.10 = \frac{1}{4}\sqrt{4N(N+1)-1} + 0.10, \quad \text{for} \quad N \geq 2.$$

These peak-values are taken as the frequencies of the <u>resonances</u> associated with the given geometry. They are close to the Regge-poles $\sqrt{\Lambda} = \frac{\rho_N}{4} - \frac{i}{4}$, i.e. to the poles of $\Gamma[\frac{1}{4} + i(\frac{\rho}{4} - \sqrt{\Lambda})]$ nearest to the real axis. Thus the two approaches agree for large N and give alternative descriptions of the same phenomenon.

We interpret a resonance as a metastable state resulting from the capture of a massive quanta by the shell of the geometry,

and this is bounded as $z \to 0$ if and only if $0 \le \operatorname{Re} \alpha \le 1$. As for the case when $c-a-b = 1-2\alpha = \pm 2m$, $m = 0,1,2,$ this is the same as $\alpha = 1/2 \mp m$, which yields unbounded solutions at $z = 0$, by the very same considerations that applied to the case $\Lambda > 0$.

The theorem is proved.

We rewrite our solution (7.26) explicitly in terms of Legendre functions by using the identity (p.562 [1])

$$F(a,a+\tfrac{1}{2},c;x) = 2^{c-1}\,\Gamma(c)(-x)^{\frac{1}{2}-\frac{1}{2}c}\,(1-x)^{\frac{1}{2}c-a-\frac{1}{2}}\,P^{1-c}_{2a-c}\,[(1-x)^{-\frac{1}{2}}]$$

valid for $-\infty < x < 0$. This yields

$$(7.39) \qquad v^{\pm}(x) = 2^{\mp 2i\sqrt{\Lambda}}\,\Gamma(1\mp 2i\sqrt{\Lambda})\cdot P^{\pm 2i\sqrt{\Lambda}}_{\alpha-1}\,(\tanh \tfrac{x}{2}).$$

Therefore, except for a scalar factor,

$$(7.40) \qquad U^{\pm} = e^{\pm 2i\mu\sqrt{\Lambda}x^4}\cdot P^{\pm 2i\sqrt{\Lambda}}_{\alpha-1}\,(\tanh \mu\, x^1),$$

with $\alpha - \frac{1}{2} \in (-\frac{1}{2},\frac{1}{2}) \cup i\mathbb{R}$ and $\Lambda \ge 0$.

Remark that when $\alpha - 1 = -\frac{1}{2} + \frac{i\rho}{2}$, with ρ real, the Legendre functions for the resonances are the so-called conical functions (p.174 [7]). The Legendre functions are given explicitly by (p.121, vol.II, [13])

$$\sqrt{\pi}\;2^{\mp 2i\sqrt{\Lambda}}\cdot P^{\pm 2i\sqrt{\Lambda}}_{\alpha-1}(x) =$$

$$= \frac{(x^2-1)^{\mp i\sqrt{\Lambda}}}{\Gamma(\pm 2i\sqrt{\Lambda}-\frac{1}{2})}\int_0^{\pi}(x+\sqrt{x^2-1}\cos\psi)^{\alpha-1\pm 2i\sqrt{\Lambda}}\cdot\sin^{\mp 2i\sqrt{\Lambda}}\psi\; d\psi,$$

valid for $\operatorname{Re} x > 0$, $\Lambda \ge 0$.

In particular when $\Lambda = 0$, $P_{\alpha-1}$ and thus U in (7.40), are just the spherical functions for the principal series (p.44,

by the same reason as in that case because we have as before $\mathrm{Re}\ b = \frac{1}{2} + \mathrm{Re}\ a > \mathrm{Re}\ a$. Only the point $z = 0$ needs consideration. We have from (7.32)

$$U = z^{\alpha} F(\tfrac{\alpha}{2}, \tfrac{\alpha}{2} + \tfrac{1}{2}, 1; 1+z^2). \tag{7.38}$$

From the identity (p. 560 [1])

$$F(a, a+\tfrac{1}{2}, c; w) = (\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1-w})^{-2a} \cdot F(2a, 2a-c+1, c; \frac{1-\sqrt{1-w}}{1+\sqrt{1-w}})$$

we get

$$U = z^{\alpha} \left(\frac{1+iz}{2}\right)^{-\alpha} F(\alpha, \alpha, 1; \frac{1-iz}{1+iz}).$$

Now when $\alpha = -m$, $m = 0,1,2,\ldots$, F is a polynomial p_m of degree m so that as $z \to 0$

$$U = z^{\alpha} p_m\left(\frac{1-iz}{1+iz}\right) \sim p_m(1) z^{-m}.$$

But since $c-a-b = 1-2\alpha = 1+2m > 0$ we get from p.556 [1], $p_m(1) = F(\alpha, \alpha, 1; 1) = \dfrac{\Gamma(1)\Gamma(1+2m)}{\Gamma(1+m)^2} \neq 0$, and therefore U is unbounded as $z \to 0$.

Similarly if $c-a = 1-\alpha = -m$, $m = 0,1,2,\ldots$ i.e., $\alpha = m+1$ we get (p.559 [1])

$$F(\alpha, \alpha, 1; \frac{1-iz}{1+iz}) = \left(\frac{2iz}{1+iz}\right)^{1-2\alpha} F(-m, -m, 1; \frac{1-iz}{1+iz})$$

so that, just as before, as $z \to 0$,

$$U \sim p_m(1) \cdot z^{1-2\alpha} = p_m(1) \cdot z^{-2m-1},$$

which is unbounded.

Therefore for the remaining cases, i.e. for $\alpha \neq \pm m$, $m = 0,1,2,\ldots$, we have $c-a-b = 1-2\alpha \neq 1 \mp 2m$. If further $c-a-b \neq \pm 2m$ then by (7.33) we have

$$U = z^{\alpha}\left(\frac{1+iz}{2}\right)^{-\alpha} \left[c_o\left(\frac{2iz}{1+iz}\right)^{1-2\alpha} + c_1\right] \qquad (c_o, c_1 \neq 0),$$

Therefore if $\frac{\alpha}{2}$ and $\frac{1-\alpha}{2} \neq -m$, $m = 0,1,2,\ldots$ we get, upon letting $z \to -i$ (i.e., by letting $\zeta \to i$ and $w = \frac{1}{1+z^2} \to \infty$) in (7.36):

$$V_\theta \sim \text{const.}\ (1+z^2)^{-\frac{\alpha}{2}}\ (1+z^2)^{\frac{\alpha}{2}}\ z^\alpha[-\ln(1+z^2)+c],$$

which is unbounded as $z \to -i$.

If $\frac{1-\alpha}{2} = -m$, $m = 0,1,2,\ldots$ then, by the above remarks,

$$\begin{aligned} V_\theta &= (1+z^2)^{-\frac{\alpha}{2}} \cdot z^\alpha \cdot \left(\frac{z^2}{1+z^2}\right)^{\frac{1}{2}-\alpha} p_m\left(\frac{1}{1+z^2}\right) \\ &= (1+z^2)^{\frac{\alpha-1}{2}}\ z^{1-\alpha}\ p_m\left(\frac{1}{1+z^2}\right) \\ &= (1+z^2)^m\ p_m\left(\frac{1}{1+z^2}\right) \cdot z^{-2m} \sim \text{const.}\ z^{-2m} \quad \text{as } z \to 0 \end{aligned}$$

and thus is unbounded at $z = 0$, unless $m = 0$. Similarly if $\frac{\alpha}{2} = -m$, $m = 0,1,2,\ldots$ then

$$V_\theta = (1+z^2)^{-\frac{\alpha}{2}} z^\alpha\ q_m\left(\frac{1}{1+z^2}\right) = (1+z^2)^m\ q_m\left(\frac{1}{1+z^2}\right) z^{-2m}$$
$$\sim \text{const.}\ z^{-2m} \quad \text{as} \quad z \to 0,$$

which is again unbounded, unless $m = 0$. In the first case, $m = 0$ means $\alpha = 1$ and in the second it means $\alpha = 0$, and these are the harmonic cases, that we explicitly ruled out, because they were already considered in Chapter VI.

To complete the proof we must show that $U = U^+ \equiv U^-$ given by (7.26) and (7.32) with $\Lambda = 0$, is C^∞ and bounded under the given assumptions on α. Therefore it will be the only bounded C^∞ solution in H, in this case.

As in the case $\Lambda > 0$ we only need to show its boundedness at $z = 0$ and at $z = \infty$. Boundedness at $z = \infty$ follows

are unbounded. Therefore boundedness at $z = 0$ requires $\alpha = \frac{1}{2} \in (-\frac{1}{2}, \frac{1}{2}) \cup i\mathbb{R}$.

As for $z \to \infty$ we observe that in (7.26) $\operatorname{Re} b = \frac{1}{2} + \operatorname{Re} a > \operatorname{Re} a$ so that from (7.30) and (7.32) we get

$$F(a,b,c;1+z^2) \sim K(z^2)^{-b} = Kz^{-(1+\alpha)\pm 2i\sqrt{\Lambda}},$$

and so

$$U^{\pm} \sim w^{\pm 2i\sqrt{\Lambda}}\, z^{-1\pm 2i\sqrt{\Lambda}}, \quad \text{as} \quad z \to \infty.$$

Therefore $U^{\pm}$ are bounded in H under the stated conditions.

Consider now the case $\Lambda = 0$. In this case the functions $V_\theta^+ = V_\theta^- \equiv V_\theta$ coincide, and so do $V_\phi^+ = V_\phi^- \equiv V_\phi$, and we get from (7.29) and (7.20)

$$(7.36) \qquad V_\theta = (1+z^2)^{-\frac{\alpha}{2}} z^\alpha F(\tfrac{\alpha}{2}, \tfrac{\alpha}{2}, \tfrac{1}{2}; \tfrac{1}{1+z^2}),$$

$$(7.37) \qquad V_\phi = (1+z^2)^{-\frac{\alpha}{2}-\frac{1}{2}} z^\alpha F(\tfrac{\alpha+1}{2}, \tfrac{\alpha+1}{2}, \tfrac{3}{2}; \tfrac{1}{1+z^2}).$$

If a, c and $c-a \neq -m$, $m = 0,1,2,\ldots$ then from

$$F(a,a,c;w) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} (-w)^{-a} \sum_{n=0}^{\infty} \frac{(a_n)(1-c+a)_n}{(n!)^2} w^{-n}[\ln(-w)+2\psi(n+1) - \psi(a+n) - \psi(c-a-n)]$$

we get, as $w \to \infty$,

$$F(a,a,c;w) \sim \text{const.}\ w^{-a}(\ln w + c), \qquad (p.560\ [1]).$$

If $a \neq -m$ but $c-a = -m$, $m = 0,1,2,\ldots$ we have (p.560 [1]) $F(a,a,a-m;w) = (1-w)^{-m-a} F(-m,-m,c;w)$, and the last factor is then a polynomial $p_m(w)$ of degree m in w. If $a = -m$, $m = 0,1,2,\ldots$ then F itself is a polynomial $q_m(w)$ of degree m in w.

as $w \to 0$, provided $c-a-b \neq \pm m$, $m = 0,1,2,\dots,$ otherwise

(7.34) $$F(a,b,a+b+m;1-w) \sim A_m - \frac{\Gamma(a+b+m)}{\Gamma(a)\Gamma(b)} w^m \ln w,$$

if $m = 0,1,2,\dots,$ and

(7.35) $$F(a,b,a+b-m;1-w) \sim \frac{\Gamma(m)\Gamma(a+b-m)}{\Gamma(a)\Gamma(b)} w^{-m} -$$
$$- (-1)^m \frac{\Gamma(a+b-m)}{\Gamma(a-m)\Gamma(b-m)} \ln w,$$

if $m = 1,2,\dots$.

Due to (7.27) we have just as before that a, b, c, $c-a$ and $c-b$ in (7.26) are never relative integers. Further $c-a-b = \frac{1}{2} - \alpha$. So if $z \to 0$, we get from (7.32), in case $\frac{1}{2} - \alpha \neq 0,\pm1,\pm2,\dots,$ $U^\pm \sim c_o\, w^{\pm 2i\sqrt{\Lambda}}\, z^\alpha (-z)^{1-2\alpha} = c_o\, w^{\pm 2i\sqrt{\Lambda}}\, z^{1-\alpha}$ if $\mathrm{Re}\,\alpha > \frac{1}{2}$, and $U^\pm \sim c_1\, w^{\pm 2i\sqrt{\Lambda}}\, z^\alpha$ if $\mathrm{Re}\,\alpha < \frac{1}{2}$ or $\alpha = \frac{1}{2} + i\rho$, $\rho \neq 0$ real.

In the first case boundedness requires $\alpha \leq 1$ an in the second $\alpha \geq 0$ (observe that α is real if $\mathrm{Re}\,\alpha \neq \frac{1}{2}$). Thus we get $0 < \alpha < 1$, $\alpha \neq \frac{1}{2}$, or $\alpha = \frac{1}{2} + i\rho$, $\rho \neq 0$, ρ real, as the boundedness condition at $z = 0$ if $\frac{1}{2} - \alpha \neq 0,\pm1,\pm2,\dots$.
If $\alpha = \frac{1}{2}$, i.e., $m = 0$ we get

$$U^\pm \sim w^{\pm 2i\sqrt{\Lambda}}\, z^{\frac{1}{2}} \ln z,$$

and if $m = \frac{1}{2} - \alpha$ $m = 1,2,\dots$ we have

$$U^\pm \sim w^{\pm 2i\sqrt{\Lambda}} \cdot [c_2 z^{\frac{1}{2}-m} + c_3\, z^{\frac{1}{2}+m} \ln z].$$

Finally if $-m = \frac{1}{2} - \alpha$, $m = 1,2,\dots$ we get

$$U^\pm \sim w^{\pm 2i\sqrt{\Lambda}} \cdot [c_4\, z^{\frac{1}{2}-m} + z^{\frac{1}{2}+m} \ln z].$$

Clearly the first case is bounded, whereas the two others

An analogous expression holds for $V_\phi^\pm$, because the additional term $1/2$ in the coefficients $\tilde{a}$, $\tilde{b}$ of ϕ in (7.20) is compensated by the power $-1/2$ of the extra factor $\sinh \frac{x}{2}$ in (7.20), cf. (7.22).

Therefore whichever the choice of $\pm$ sign in (7.31), the corresponding function behaves as $c(z+i)^{\pm 2i\sqrt{\Lambda}}$ + const., $c \neq 0$, which is not even continuous at $z = -i$, i.e., at $\zeta = i \in H$. The same holds for $V_\phi^\pm$.

The only way out is that some linear combination of V_θ^+ and V_ϕ^+, and of V_θ^- and V_ϕ^-, will cancel this singularity. Indeed, if we introduce the notation (7.28) in (7.26) we get, except for a scalar factor,

$$U^\pm = w^{\pm 2i\sqrt{\Lambda}}\, z^\alpha F(a,b,c;1+z^2), \tag{7.32}$$

with a, b, c being the constants in (7.26). In this expression the singularity at $z = -i$ has simply been removed. Therefore $U^\pm$ are C^∞ in H, because they are real analytic in H.

The functions $v^\pm(x)$ in (7.25') are solutions of (7.17) and indeed $v^+(v^-)$ is the only solution belonging to $L^2(0,\infty)$ when $\operatorname{Im}\sqrt{\Lambda} > 0$ $(\operatorname{Im}\sqrt{\Lambda} < 0)$, respectively, (see p.103 [14]). Notice that $(0,\infty)$ corresponds to $r > r_0$ in concrete space.

To check boundedness in H it suffices to check boundedness at the remaining singularities $z = 0$ and $z = \infty$ which lie in ∂H (together with $w = \infty$). For that purpose we observe that if a, b, c, $c-a$ and $c-b$ are distinct from $0,-1,-2,\ldots$ then (pp.559, 560 [1]):

$$F(a,b,c;1-w) \sim \frac{\Gamma(a+b-c)\Gamma(c)}{\Gamma(a)\Gamma(b)}\, w^{c-a-b} + \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}, \tag{7.33}$$

The possible singular points of (7.29) as a function of z are the points $z = 0$, $z = \infty$, $z = -i$, which correspond to $\eta/\xi + i = 0$, to $\eta = \infty$, and to $\xi = 0$ and $\eta = 1$, i.e. $\zeta = i$, respectively. The point $\zeta = i$ belongs in H and the two others in ∂H.

Consider the case when $\Lambda > 0$.

From (7.29) and (7.19) we have

$$\tilde{c}-\tilde{a} = \left(\frac{1}{2} - \frac{\alpha}{2} + i\sqrt{\Lambda}\right), \qquad \tilde{b}-\tilde{a} = -2i\sqrt{\Lambda},$$

$$\tilde{c}-\tilde{b} = \left(\frac{1}{2} - \frac{\alpha}{2} - i\sqrt{\Lambda}\right),$$

and these are never relative integers if $\Lambda > 0$, by (7.27).

Similarly for ϕ in (7.20)

$$\tilde{c}-\tilde{a} = 1 - \frac{\alpha}{2} + i\sqrt{\Lambda}, \qquad \tilde{b}-\tilde{a} = -2i\sqrt{\Lambda}$$

$$\tilde{c}-\tilde{b} = 1 - \frac{\alpha}{2} - i\sqrt{\Lambda},$$

which again are never relative integers. By the same token neither are $\tilde{a}$, $\tilde{b}$ in (7.19) and (7.20).

This means that for the values of $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$ in (7.19) and (7.20) we have (p.559 [1]), as $w \to \infty$:

$$(7.30) \qquad F(\tilde{a},\tilde{b},\tilde{c};w) \sim \frac{\Gamma(\tilde{c})\Gamma(\tilde{b}-\tilde{a})}{\Gamma(\tilde{b})\Gamma(\tilde{c}-\tilde{a})}(-w)^{-\tilde{a}} + \frac{\Gamma(\tilde{c})\Gamma(\tilde{a}-\tilde{b})}{\Gamma(\tilde{b})\Gamma(\tilde{c}-\tilde{b})}(-w)^{-\tilde{b}},$$

$$= c_1 w^{-\tilde{a}} + c_2 w^{-\tilde{b}}, \qquad (c_1, c_2 \neq 0).$$

Using this in (7.29), when $z \to -i$, we get

$$(7.31) \qquad V_\theta^\pm \sim w^{\pm 2i\sqrt{\Lambda}}\left[c_1(z+i)^{\left\{\begin{smallmatrix}2i\sqrt{\Lambda}\\ 0\end{smallmatrix}\right\}} + c_2(z+i)^{\left\{\begin{smallmatrix}0\\ -2i\sqrt{\Lambda}\end{smallmatrix}\right\}}\right],$$

respectively.

7.2.2 THEOREM. <u>The only non-harmonic periodic solutions of (7.7) of the form (7.8), which are</u> C^∞ <u>and bounded when continued into the Poincaré half-plane</u> H <u>defined above, are the functions</u>

$$(7.26) \quad U^\pm = e^{\pm\sqrt{p}X^4} \cdot v^\pm(x)$$
$$= e^{\pm 2\mu i\sqrt{\Lambda}X^4} \cdot \frac{\cosh^\alpha(\frac{x}{2})}{\sinh^{\alpha\mp 2i\sqrt{\Lambda}}(\frac{x}{2})} \cdot F\left(\tfrac{1}{2}\alpha\mp i\sqrt{\Lambda}, \tfrac{1}{2}(\alpha+1)\mp i\sqrt{\Lambda}, 1\mp 2i\sqrt{\Lambda}; -\frac{1}{\sinh^2(\frac{x}{2})}\right)$$

where $\Lambda \geq 0$ and

$$\alpha - \tfrac{1}{2} \in (-\tfrac{1}{2}, \tfrac{1}{2}) \cup i\mathbb{R}.$$

(The same statement holds for the lower Poincaré half-plane, with the choice $\gamma = -1$).

<u>Proof</u>: According to (7.4) non-harmonicity means $k \neq 0$, and so by (7.15), $\alpha \neq 0$ and $\alpha \neq 1$. On the other hand as k is real, (7.15) yields that

$$(7.27) \qquad \alpha - \tfrac{1}{2} \in i\mathbb{R} \cup \mathbb{R} .$$

Periodicity means that $\Lambda \geq 0$ (by (7.11) and (7.16)).

To check the behavior in H set $\gamma = 1$ in (7.25) and rewrite it with the new variables

$$(7.28) \qquad w = \xi + i, \qquad z = \frac{\eta}{\xi+i} ,$$

so that, except for a scalar factor,

$$(7.29) \qquad V_\theta^\pm = w^{\pm 2i\sqrt{\Lambda}}(1+z^2)^{\pm i\sqrt{\Lambda}-\frac{\alpha}{2}} \cdot z^\alpha F\left(\tilde{a},\tilde{b},\tilde{c};\frac{1}{1+z^2}\right)$$

A similar expression is valid for $V_\phi^\pm$.

Hence in terms of (ξ,η) coordinates, we have

$$(7.24)\quad v_\theta^\pm = (\xi-\gamma)^{\pm 2i\sqrt{\Lambda}}\left[\left(\frac{\eta}{\xi-\gamma}\right)^2-1\right]^{\pm i\sqrt{\Lambda}-\frac{\alpha}{2}}\left(\frac{\eta}{\xi-\gamma}\right)^\alpha F\left(\tilde{a},\tilde{b},\tilde{c};-\left[\left(\frac{\eta}{\xi-\gamma}\right)^2-1\right]^{-1}\right),$$

with $\tilde{a}$, $\tilde{b}$, $\tilde{c}$ being the corresponding parameters in (7.19).

By the complexification leading to (6.42) this becomes, except for a scalar factor,

$$(7.25)\qquad v_\theta^\pm = (\xi+i\gamma)^{\pm 2i\sqrt{\Lambda}}\left[1+\left(\frac{\eta}{\xi+i\gamma}\right)\right]^{\pm i\sqrt{\Lambda}-\alpha/2}\left(\frac{\eta}{\xi+i\gamma}\right)^\alpha F\left(\tilde{a},\tilde{b},\tilde{c};\frac{1}{1+\left(\frac{\eta}{\xi+i\gamma}\right)^2}\right)$$

Since $v_\theta^\pm$ split into the product of functions of $(\xi-\gamma)$ and $\frac{\eta}{\xi-\gamma}$ in (7.24), it is clear that if we require $v_\theta^\pm$ to be C^∞ in a Poincaré-like half-plane H then, in the case $\Lambda \neq 0$, we must have $\xi-\gamma \neq 0$ there. If we require γ to be real then the appropriate complexification is $\xi \mapsto i\xi$ and $\eta \mapsto \eta$, just as in Section 6.7, with $\gamma \neq 0$. Here again we have two choices $\gamma > 0$, say $\gamma = 1$, (upper half-plane) and $\gamma < 0$, say $\gamma = -1$, for the lower half-plane. From now on we fix $\gamma = 1$.

An expression similar to (7.22) holds for $v_\phi^\pm$ except that $\tilde{a}$, $\tilde{b}$, $\tilde{c}$ have the values given in (7.20) and there is an extra factor $\left[\left(\frac{\eta}{\xi+i\gamma}\right)^2+1\right]^{-1/2}$.

For later reference we single out the following solutions of (7.17)

$$(7.25')\qquad v^\pm(x) = \frac{\Gamma(\frac{1}{2})\Gamma(1\mp 2i\sqrt{\Lambda})}{\Gamma(1-\frac{1}{2}\alpha\mp i\sqrt{\Lambda})\Gamma(\frac{1}{2}+\frac{1}{2}\mp i\sqrt{\Lambda})}\theta(x) - \frac{\frac{1}{2}\Gamma(-\frac{1}{2})\Gamma(1\mp 2i\sqrt{\Lambda})}{\Gamma(\frac{1}{2}-\frac{1}{2}\alpha\mp i\sqrt{\Lambda})\Gamma(\frac{1}{2}\alpha\mp i\sqrt{\Lambda})}\phi(x).$$

We now prove the following important

the x-interval may be taken as the whole real line $(-\infty,\infty)$ as far as spectral theory is concerned (p. 103 [14]). That is, in our case, we may consider the problem as defined in the wormhole topology 5.13, with $x < 0$ corresponding to the region inside the shell of the particle together with its continuation into anti-space across the center $r = 0$.

Now any solution of (7.17) is a linear combination of the solutions (p.103 [14]),

$$(7.19) \quad \theta(x) = \cosh^{\alpha} \tfrac{1}{2}x \cdot F(\tfrac{1}{2}\alpha + i\sqrt{\Lambda},\ \tfrac{1}{2}\alpha - i\sqrt{\Lambda},\ \tfrac{1}{2};\ -\sinh^2 \tfrac{x}{2}),$$

and

$$(7.20) \quad \phi(x) = \cosh^{\alpha} \tfrac{x}{2} \cdot \sinh \tfrac{x}{2} \cdot F(\tfrac{1}{2}+\tfrac{1}{2}\alpha+i\sqrt{\Lambda},\ \tfrac{1}{2}+\tfrac{1}{2}\alpha-i\sqrt{\Lambda},\ \tfrac{3}{2};\ -\sinh^2 \tfrac{x}{2}),$$

and these satisfy

$$(7.20') \qquad \theta(0) = \phi'(0) = 1, \qquad \theta'(0) = \phi(0) = 0.$$

By (7.11) and (7.16) the solutions (7.8) are linear combinations of

$$V_{\theta}^{\pm} = e^{\pm 2\mu i\sqrt{\Lambda}X^4} \cdot \theta \quad \text{and} \quad V_{\phi}^{\pm} = e^{\pm 2\mu i\sqrt{\Lambda}X^4} \cdot \phi,$$

corresponding to the same choice of $\pm$ sign, vertically ordered.

From (6.11) and (6.12), and (7.12), one gets

$$(7.21) \qquad e^{\pm\sqrt{p}X^4} = [\eta^2 - (\xi-\gamma)^2]^{\pm i\sqrt{\Lambda}},$$

$$(7.22) \qquad \sinh \tfrac{x}{2} = [(\tfrac{\eta}{\xi-\gamma})^2 - 1]^{-1/2},$$

$$(7.23) \qquad \cosh^{\alpha} \tfrac{x}{2} = \left[\frac{(\frac{\eta}{\xi-\gamma})^2}{(\frac{\eta}{\xi-\gamma})^2 - 1}\right]^{\alpha/2},$$

$$(7.23') \qquad \tanh \tfrac{x}{2} = \frac{\xi-\gamma}{\eta}.$$

(Remark this Λ has nothing to do with the cosmological constant in Chapter V.)

From now on we agree to represent any complex number Λ in the form $\Lambda = |\Lambda|\, e^{i \arg \Lambda}$ with $0 \le \arg \Lambda < 2\pi$, so that $\sqrt{\Lambda} = |\Lambda|^{1/2}\, e^{\frac{i}{2} \arg \Lambda}$ satisfies $\operatorname{Im} \sqrt{\Lambda} \ge 0$. The other root will be $-\sqrt{\Lambda}$.

7.2.1 POINCARÉ HALF-PLANE

Recall that we have

$$\tanh \frac{x}{2} = \tanh \mu X^1 = \operatorname{sgn} Q \cdot e^{-\mu|r-r_o|},$$

where Q is the charge of the geometry. To fix ideas we consider the case $Q > 0$, so that then $x \ge 0$, and so $x = +\infty$ corresponds to $r = r_o$ and $x = 0$ to $r = +\infty$.

By the change of coordinates (6.39) and further by complexification, the eigenvalue equation (7.7) goes into

$$-\eta^2\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right)U = \frac{k}{\mu^2}\,U, \tag{7.18}$$

in the Poincaré half-plane.

As mentioned before, when $k \neq 0$ (non-harmonic state), U can only correspond to a state in the even principal or in the supplementary series representation of $SL_2(R)$. In this case it is necessary that U be a C^∞ bounded function on $\eta > 0$. So we will check this condition for U having the particular form (7.8); in this case the corresponding v satisfies (7.17) in the half-line $x \ge 0$.

Since the differential equation (7.17) is regular at the end-point $x = 0$ and since the potential is an even function of x,

$$\cosh^2 \mu X^1(\partial_1^2-\partial_4^2)U = kU, \tag{7.7}$$

(according to (7.1)), which admit the separation of variables

$$U = u(X^1)\cdot T(X^4), \tag{7.8}$$

we get

$$T'' = pT \tag{7.9}$$

with p a real constant, and

$$u'' - (k \operatorname{sech}^2 \mu X^1 + p)u = 0. \tag{7.10}$$

Therefore, from (7.9), $T(X^4)$ consists of linear combinations of

$$e^{\pm\sqrt{p}\, X^4}. \tag{7.11}$$

If we set in (7.10)

$$\mu X^1 = x/2, \tag{7.12}$$

$$v(x) = u(X^1) = u(x/2\mu), \tag{7.13}$$

we obtain

$$v'' - (\frac{p}{4\mu^2} + \frac{k}{4\mu^2} \operatorname{sech}^2 \frac{x}{2})v = 0. \tag{7.14}$$

(This x should not be confounded with x in Chapter V).

In order to conform with the notation in [14], we set further

$$k/\mu^2 = \alpha(1-\alpha), \tag{7.15}$$

$$\Lambda = -p/4\mu^2, \tag{7.16}$$

and we get

$$v'' + (\Lambda - \frac{\alpha(1-\alpha)}{4 \cosh^2 \frac{x}{2}})v = 0. \tag{7.17}$$

$$(7.1) \qquad \Delta_2 = \cosh^2 \mu x^1 [\partial_1^2 - \partial_4^2] + \partial_2^2 + \partial_3^2 \equiv \Delta_r + \Delta_c \, ,$$

where Δ_r refers to (x^1,x^4)-space and Δ_c to (x^2,x^3).

Assume that the state Φ satisfying $\Delta_2 \Phi = 0$ admits the separation of variables

$$(7.2) \qquad \Phi = U(x^1,x^4) \cdot W(x^2,x^3),$$

so that we have then

$$(7.3) \qquad \Delta_r U = kU,$$

$$(7.4) \qquad \Delta_c W = -kW.$$

We expect the operators to be essentially self-adjoint, so the constant k should be real.

If Φ is non-harmonic then $k \neq 0$ in (7.4) and so comparing (7.3) with (6.28) and (6.30), and with $\Delta_r \varphi = 0$ in the discrete case, we reach the above mentioned conclusion.

Let (σ,θ) denote polar coordinates in (x^2,x^3)-space. The solutions of (7.4) of the form $W = S(\sigma)F(\theta)$, that are regular at $\sigma = 0$, are given by

$$(7.5) \qquad W_n = J_n(\sqrt{k}\,\sigma) e^{\pm in\theta}, \qquad n = 0,1,2,\ldots$$

where

$$(7.6) \qquad J_n(s) = \sum_{p=0}^{\infty} \frac{(-1)^p (s/2)^{n+2p}}{p!\,\Gamma(n+p+1)}$$

is the Bessel function of the first kind. (Here, when $k < 0$, we set $\sqrt{k} = i\sqrt{|k|}$).

Similarly, if we look for solutions of (7.3), that is, of

which, depending on the imparting energy, goes into an internally excited state plus an external resonant state; for maximum stability, the internal state, together with the captured massive quanta, should then constitute a system with the same energy as that of the most stable internally excited state of the geometry, or a state as close to it as possible. The total energy of such system is then given by

$$E_o + E_o' \{n' + \left[1 + \left(\frac{2n_o+1}{1+|s_o|}\right)^2 \Lambda_N\right]^{1/2} \}, \qquad N = 1,2,...,$$

with $n' \leq n_o-1$, which is the same as (7.0), once we make use of (6.0).

We now present the detailed analysis leading to these results.

7.2 NON-HARMONIC STATES

We consider states in the envelope space $\hat{\mathcal{E}}$ of the algebra generated by the monochromatic algebras $\mathfrak{h}$, $\mathfrak{h}'$, $\mathfrak{h}$ and $\bar{\mathfrak{h}}'$ defined in 5.10, that is of the larger algebra where also algebraic combinations of both analytic and anti-analytic functions of $y+iz$ are allowed. In this case a state $\Phi(X^1,X^2,X^3,X^4)$ need no longer be an harmonic function of X^2, X^3.

If a non-harmonic state Φ of the form (7.2) below, with $\Delta_2\Phi = 0$, should correspond in the Poincaré half-plane to an eigenstate of the invariant Laplace operator associated with an irreducible unitary representation of $SL_2(\mathbb{R})$, then it can only belong in the principal or in the supplementary series, as given in 6.5.3.

Indeed, the Laplace-Beltrami operator for the metric (5.20) (with $B=1$), is

[5]).

7.2.3 REMARKS

1. It is clear that the same result holds if we take $\gamma = -1$ and work in the lower Poincaré half-plane H^-: $\zeta = \xi + i\eta$, $\eta < 0$ (or equivalently on H with $\bar{\zeta} = \xi - i\eta$), for all the formulae remain valid under this change.

2. Obviously all choices $\gamma > 0$ ($\gamma < 0$) are equivalent to $\gamma = 1$ ($\gamma = -1$, resp.) because we can map the vertex $\zeta = (\gamma, 0)$ into $(1,0)$ (resp. $(\gamma,0)$ into $(-1,0)$) by a dilatation $\zeta \mapsto \zeta/|\gamma|$ in $SL_2(\mathbb{R})$, which amounts to a simple change of time origin.

3. From 8.1.2 [1] we have

$$P_{\alpha-1}^{\pm 2i\sqrt{\Lambda}}(\tanh\mu\, X^1) \sim \frac{1}{\Gamma(1 \mp 2i\sqrt{\Lambda})}\, e^{\pm 2\mu i\sqrt{\Lambda} X^1} \quad \text{at} \quad X^1 \sim \infty,$$

i.e. at $r \sim r_o$. Thus $U^{\pm}$ behave as $e^{\pm 2i\mu\sqrt{\Lambda}(X^4 + X^1)}$ at the outside of the shell of the particle. By (6.42) these plane waves correspond to $(\bar{\zeta} + i\gamma)^{\pm 2i\sqrt{\Lambda}}$. Consequently $\bar{U}^{\pm}$ at r_o correspond to $(\zeta - i\gamma)^{\mp 2i\sqrt{\Lambda}}$. Thus if $\gamma = 1$, $U^{\pm}$ are regular in H, while $\bar{U}^{\pm}$ are regular in H^-. The converse holds if $\gamma = -1$. Since U and $\bar{U}$ are both bounded solutions of the same pair of equations (7.9) and (7.10), (but with reversed signs for $\sqrt{\Lambda}$) it follows that $\gamma = 1$ and $\gamma = -1$ provide the same sets of solutions in H, and in H^-. Thus the two choices $\gamma = 1$ and $\gamma = -1$ are equivalent. (This is related to the fact that $s = i\rho$ and $s = -i\rho$ yield equivalent representations of the principal series.) Notice that the same argument fails in the case of the discrete series of representa-

tions, for $\left(\frac{1}{\zeta+i}\right)^{\frac{m}{\mu}}$ is bounded and C^∞ in H whereas its inverse $(\zeta+i)^{\frac{m}{\mu}}$ is unbounded.

7.2.4 STANDING SOLUTIONS. MIXING.

We have just seen that at the shell of the particle, $U^\pm$ behave as $e^{\pm 2i\mu\sqrt{\Lambda}(X^4+X^1)}$. To build out of these two solutions a solution that is a standing wave at $r = r_o$ we must again use two geometries with opposite charges so that X^4+X^1 be an outgoing wave for one and an incoming wave for the other. At the same time we must change the sign of $\sqrt{\Lambda}$ in one of them so as to be able to factor out the time exponential. Let then denote by F the group of transformations $\sqrt{\Lambda} \to -\sqrt{\Lambda}$, i.e. the group of rotations by $\pm$ 180° in the complex $\sqrt{\Lambda}$-plane.

Write $q = \operatorname{sgn} Q$, where Q is the charge of the geometry, and $f = \operatorname{sgn} \operatorname{Im} \sqrt{\Lambda}$, so that $f = 1$ corresponds to the usual construction of the spectral theory for the Schrödinger operator in $\operatorname{Im} \sqrt{\Lambda} > 0$ and $f = -1$ to the (isomorphic) construction in $\operatorname{Im} \sqrt{\Lambda} < 0$. The functions $\mathfrak{s}^\pm_{f,q} = e^{\pm 2if\mu\sqrt{\Lambda}(X^4+q|X^1|)}$, $\sqrt{\Lambda} \geq 0$, represent then all possible choices for the given pair of waves (but in fact only four of them are distinct, say $\mathfrak{s}^+_{f,q}$). If we take $\mathfrak{s}^+_{1,1} = e^{2i\mu\sqrt{\Lambda}(X^4+|X^1|)}$, we must take $\mathfrak{s}^-_{-1,-1} = $
$= e^{2i\mu\sqrt{\Lambda}(X^4-|X^1|)}$ so that

$$\mathfrak{s}^+_{1,1} - \mathfrak{s}^-_{-1,-1} = 2ie^{2i\mu\sqrt{\Lambda}X^4}\sin 2\mu\sqrt{\Lambda}|X^1| \tag{7.40'}$$

yields a standing wave at the shell.

In the general case standing solutions are generated by the pair $\mathfrak{s}^+_{f,q}$ and $\mathfrak{s}^-_{-f,-q}$. Thus the discontinuous groups F (frequency conjugation) and C (charge conjugation) are always

mixed in the standing solutions. The total symmetry group generated quantum-mechanically on the corresponding four-dimensional space is thus SU_4.

Since time-inversion T is isomorphic to FC, in view of the fact that $\sqrt{\Lambda}(X^4+X^1) \xmapsto{FC} \sqrt{\Lambda}(-X^4+X^1)$, we may say, alternatively, that T and C are mixed in SU_4.

This should be compared with the case of the discrete series representation when it is C and P that are mixed in SU_4. Here P is parity-conjugation, which is isomorphic to the discontinuous subgroup corresponding to the fiber S^1, or to the polarization-orientation subgroup π.

It is important to notice that T is only isomorphic to FC. In other words FC just mimicks the action of T. This is relevant because, while we can change the sign of the charge of the geometry and have both coexist, we cannot have geometries with oppositely oriented time-arrows in the same space-time, corresponding to the same particle (however they may do so in space and anti-space or when they correspond to particle and anti-particle). Therefore whereas $X^4 \mapsto -X^4$ should be considered as non-performable if it is to produce a new time-oriented geometry coexisting with the original one and for the same particle, the operations $\sqrt{\Lambda} \to -\sqrt{\Lambda}$ and $X^1 \mapsto -X^1$ can be performed and produce the same effect on $U^{\pm}$ as T.

7.3 NON-HARMONIC STATES AS EIGENSTATES OF UNITARY REPRESENTATIONS

We have just shown that the periodic non-harmonic states of the form $u(X^1)T(X^4)W(X^2,X^3)$ upon complexification become bounded C^{∞} functions in the Poincaré half-plane H if and

only if they are of the form (7.26) with $\Lambda \geq 0$ and $\alpha - \frac{1}{2} \in (-\frac{1}{2}, \frac{1}{2}) \cup i\mathbb{R}$.

Comparing (7.3) (and (7.15)) with (6.28) we get

$$\frac{1-s^2}{4} = \frac{k}{\mu^2} = \alpha(1-\alpha) \tag{7.41}$$

(7.42) i.e.

$$\alpha - \frac{1}{2} = \pm \frac{s}{2}.$$

The above condition then becomes $s \in (-1,1) \cup i\mathbb{R}$.

Now $s \in \mathbb{R}$ with $0 < |s| < 1$ is precisely the condition that defines the supplementary series representation, whereas $s \in i\mathbb{R}$ is the condition for the even principal series.

Thus if we look for non-harmonic periodic solutions separating in the variables X^1 and X^4, which when complexified satisfy the requirements of smoothness and boundedness required of states in the non-discrete series of unitary representations, then we get just the two states $U^{\pm}$ in (7.26) with $\Lambda > 0$ (or U if $\Lambda = 0$), and with $\alpha - \frac{1}{2} = \frac{s}{2}$, where s is the scalar indexing the unitary representation.

In Chapter VIII we show that, when $s \in i\mathbb{R}$, the states $U^{\pm}$, after complexification, are indeed appropriate limits of eigenstates of the principal series representations under the time-translation fiber group $\mathcal{S}$, and, in fact, the only ones with such property. Thus they belong in fact to a certain minimally extended representation.

7.4 SPECTRAL PROPERTIES

From (7.40) we obtained that, at $X^1 = \infty$ (i.e. $r = r_o$), $U^{\pm} \sim e^{\pm 2i\mu\sqrt{\Lambda}(X^4+X^1)}$, whereas at $X^1 = 0$, (i.e. $r = \infty$), we

get that $U^{\pm} \sim$ const. $e^{\pm 2i\mu\sqrt{\Lambda}x^4}$, because $P^{\pm 2i\sqrt{\Lambda}}_{-\frac{1}{2}+\frac{i\rho}{2}}(0) \neq 0$, in view of 8.6.1 in [1].

Thus $U^{\pm}$ are not bound states, i.e. they do not belong in $L_2(R)$. In fact they are free states and so describe scattering waves of the operator in (7.7). Indeed, we now show that the spectrum of the operator (7.17) consists of the real semi-axis $\Lambda \geq 0$ and is purely continuous.

First notice that, from our previous construction of $U^{\pm}$ as linear combinations of $V^{\pm}_{\theta}$ and $V^{\pm}_{\phi}$ with non-zero coefficients and from (7.20') we see that neither $U^{\pm}$ nor its derivatives are zero at $x = 0$, i.e. there are no particular homogeneous boundary conditions imposed at $x = 0$. Since the potential function in (7.17) is even in x, the spectral properties of (7.17) are the same whether we consider it on $(0,\infty)$ or on $(-\infty,\infty)$ (pp.38,58 [14]). This second choice corresponds to continuation of our metric by symmetry into anti-space, i.e. to the wormhole topology (see 5.13).

The spectrum of (7.17) consists in general of the continuous spectrum $\Lambda \geq 0$ plus a finite discrete spectrum in $\Lambda < 0$ (p.97 [14]).

In the expansion of functions $f \in L_2(-\infty,\infty)$ the continuous spectrum contributes

$$(7.43) \quad \frac{1}{\pi}\int_0^{\infty} \theta(x,\Lambda)\xi'(\Lambda)d\Lambda \int_{-\infty}^{\infty} \theta(y,\Lambda)f(y)dy + \frac{1}{\pi}\int_0^{\infty} \phi(x,\Lambda)\zeta'(\Lambda)d\Lambda \int_{-\infty}^{\infty} \phi(y,\Lambda)f(y)dy,$$

where

$$(7.44) \qquad \xi'(\Lambda) = \frac{1}{2}\,\mathrm{Im}\left(\frac{1}{m_2(\Lambda)}\right), \qquad \zeta'(\Lambda) = -\frac{1}{2}\,\mathrm{Im}(m_2(\Lambda)),$$

$$(7.45) \qquad \frac{1}{m_2(\Lambda)} = \frac{\Gamma(\frac{1}{2}-\frac{1}{2}\alpha-i\sqrt{\Lambda})\Gamma(\frac{1}{2}\alpha-i\sqrt{\Lambda})}{\Gamma(1-\frac{1}{2}\alpha-i\sqrt{\Lambda})\Gamma(\frac{1}{2}+\frac{1}{2}\alpha-i\sqrt{\Lambda})} .$$

The discrete part of the spectrum is given by the zeros and poles of $m_2(\Lambda)$ on the real axis and contributes according to the residues of m_2 and m_2^{-1}.

Thus the zeros of $m_2(\Lambda)$ are given by

$$\frac{1}{2} - \frac{\alpha}{2} - i\sqrt{\Lambda} = -n \qquad \text{or}$$

$$\frac{\alpha}{2} - i\sqrt{\Lambda} = -n ,$$

while its poles are

$$1 - \frac{\alpha}{2} - i\sqrt{\Lambda} = -n \qquad \text{or}$$

$$\frac{1}{2} + \frac{\alpha}{2} - i\sqrt{\Lambda} = -n , \qquad n = 0,1,2,\ldots$$

Using (7.42) these relations become

$$\begin{aligned} \pm s &= 4n+1 - 4i\sqrt{\Lambda} \\ &= -4n-1 + 4i\sqrt{\Lambda} \\ &= 4n+3 - 4i\sqrt{\Lambda} \\ &= -4n-1 + 4i\sqrt{\Lambda}, \qquad n = 0,1,2,\ldots, \quad \text{respectively.} \end{aligned}$$

When $\Lambda > 0$ it is thus seen that $s \notin (-1,1) \cup i\mathbb{R}$, for the above values s, while if $\Lambda \le 0$ then $\sqrt{\Lambda} = i\sqrt{|\Lambda|}$ and $-i\sqrt{\Lambda} = \sqrt{|\Lambda|} \ge 0$ so that the above $s \in (-\infty,-1] \cup [1,\infty)$. Thus the discrete spectrum is void, as claimed.

7.5 REFLECTIONLESS CONDITION. SINGULAR METRIC SOLUTIONS

We next determine which values of $s \in (-1,1) \cup iR$ are physically relevant. A physically important condition is for the potential $-\frac{k}{\mu^2}\,\mathrm{sech}^2\mu\rho$ in (7.10) to be <u>reflectionless</u>.

This means that plane waves coming from $-\infty$ in x-space remain unchanged in shape at $x = +\infty$. Since $e^{-\mu|r-r_0|} = \tanh \frac{x}{2}$, the condition $x = +\infty$ means $r = r_0$ and $x = -\infty$ means also $r = r_0$, but coming from inside the shell. That is, the reflectionless condition simply states that the shell of the particle is totally transparent (except for a possible shift in phase). However this requires $\frac{k}{\mu^2} = -n(n+1)$, with $n \in \mathbb{N}$ (p.435[4]), whereas in our case $\frac{k}{\mu^2} = \frac{1-s^2}{4} > 0$, since $s \in (-1,1) \cup i\mathbb{R}$.

Consequently the reflectionless condition cannot be applied directly to define resonances. Yet this condition enters indirectly, as we now show.

Indeed, we have neglected one solution of the Liouville equation

$$\partial_1^2 \log a = 4\lambda a^{-1},$$

obtained by taking a^{-1} time independent in (5.18), for we have only considered real coefficients B, C in the solution (5.19). There is however a new real-valued singular solution obtained by setting $B = 1$ and $C = i\frac{\pi}{2\mu}$, namely

$$a^{-1} = -\mathrm{cosech}^2 \mu X^1. \tag{7.46}$$

This is a perfectly valid solution in (4.1), provided we take now $\alpha^{-1} = -1$ instead of $\alpha^{-1} = 1$, so as to get

$$ds^2 = -\mathrm{cosech}^2 \mu\rho[d\rho^2 - dt^2] - dy^2 - dz^2, \tag{7.47}$$

and thus retain the time-character of t.

This solution has a double-pole singularity at the origin $\rho = 0$ (i.e. at $r = \infty$) and it decays exponentially at $\rho = \infty$

(i.e. at $r = r_o$). Its spectral properties are well-known and, in fact, both potentials are special cases of a more general one (p. 92[13]).

We now show that this second solution is simply what one obtains if one takes the analytic continuation of the (r,t) part of the standard metric (5.39) into the region $r-r_o < 0$, i.e. across the shell of the geometry. In contradistinction to the original continuation by symmetry, which yields anti-space, we can call the new space so obtained, the analytic anti-space. It is clear that now there is need for the discontinuity in (y,z)-space which consists in changing α^{-1} from 1 into -1 to keep the signature appropriate.

In fact analytic continuation of

$$d\sigma^2 = \frac{dr^2}{e^{2\mu(r-r_o)}-1} - (1-e^{-2\mu(r-r_o)})dt^2 \tag{7.48}$$

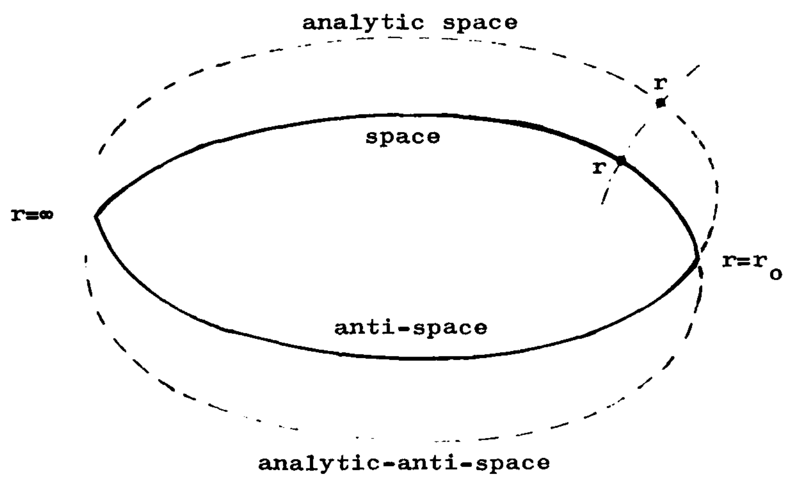
from $r-r_o > 0$ into $r-r_o < 0$ amounts to change of μ into $-\mu$ in (5.0), which gives

$$d\sigma^2 = -\left[\frac{dr^2}{(1-e^{-2\mu|r-r_o|})} - (e^{2\mu|r-r_o|}-1)dt^2\right] \tag{7.49}$$

and this is just the same as exchanging the coefficients after taking their reciprocals, and changing sign. Changing sign of $dy^2 + dz^2$ too and trying to obtain the isotropic form (5.21) with coefficient $\varphi(\rho)$, say, yields precisely the same equation as it would if we applied the same procedure to (7.48) instead, in view of the above properties of the coefficients. Thus we get the same solution $e^{-\mu|r-r_o|} = \tanh\mu\rho$, $\rho > 0$, and this yields the coefficient $\operatorname{sech}^2\mu\rho$ for $-dt^2$ in (7.48) and

$-\mathrm{cosech}^2 \mu\rho$ for $-dt^2$ in (7.49), in turn. Therefore the singular solution (7.47) can be thought of as being the analytic continuation of the original solution into $r - r_o < 0$, plus the change of sign of $dy^2 + dz^2$. This is the same as changing μ into $-\mu$ in (5.0) and the sign of $dy^2 + dz^2$.

Conversely we may continue analytically the standard metric given in anti-space, into concrete space thus yielding what we shall call analytic space, thereby ascribing to them a completely symmetric role. Graphically we represent this by the figure below in terms of the r-coordinate alone:



Therefore the more general situation originated by the existence of these two solutions should be represented by the statistical superposition of space and analytic space and, correspondingly of anti-space and analytic anti-space. Remark that having smooth continuation in (r,t) across the shell implies the discontinuity in $dy^2 + dz^2$ (change of coefficient from $+1$ to -1) whereas the *non-smooth* continuation by sym-

metry is in turn smooth in the component $dy^2 + dz^2$. The figure is purposely drawn so as to show smoothness and non-smoothness of the continuations.

Now if we want to superpose states $\Delta_2\varphi = 0$ which are expressed by the same function factor in the (y,z) variables, and thus depend on the same coupling constant k, we notice that if the potential is $\operatorname{sech}^2\mu\rho$ for concrete space it will be $\operatorname{cosech}^2\mu\rho$ for analytic space because on coming from anti-space $\operatorname{sech}^2\mu\rho$ becomes $-\operatorname{cosech}^2\mu\rho$ but since $dy^2 + dz^2$ also changes sign this is the same as ending up with $\operatorname{cosech}^2\mu$ and $dy^2 + dz^2$ in analytic space. Thus if $k \neq 0$ we can always require that either one or the other be reflectionless, and this implies the general condition

$$(7.50) \qquad |k| = \mu^2 N(N+1) \qquad N \in \mathbb{N} \quad \text{if} \quad k \neq 0,$$

to guarantee reflectionless either is space or in analytic space.

Remark that we have thus established a duality between resonances $(k > 0)$ and reflectionless states with $k < 0$, by analytic continuation of the metric from space into anti-space; this puts them on an equal footing. (In fact this is at the root of the Cooper pairing affair, Chapter X).

As mentioned in 7.1, these pairs of solutions seem to be characteristic of resonance phenomena; see for instance p in [11]. They are characteristic of non-harmonic states, because the pairing between (r,t) and (y,z) spaces due to the non-zero constant k, implies also the pairing $-k$ in the alternative metric solution. In fact we will show in Chapter X that the statistical superposition of the two metric solutions makes the

average metric conformally flat at infinity thus removing in particular the parabolic degeneracy of each of these metrics along their spin-axis when taken separately. Thus they come into play not only because they are possible solutions but also because their superposition has the proper boundary condition at infinity.

From (7.41) and (7.50) we thus get $\alpha(1-\alpha) = \frac{1-s^2}{4} = N(N+1)$ and since $s = i\rho$,

(7.51) $$\rho = \rho_N = \sqrt{4N(N+1)-1}, \qquad N = 1,2,\ldots .$$

Since $i\rho$ and $-i\rho$ yield equivalent representations (which is also clear from the fact that $P^{\mu}_{\nu}(z) \equiv P^{\mu}_{-\nu-1}(z)$ in general plus the fact that in our case $\nu = \alpha - 1 = -\frac{1}{2} - i\frac{\rho}{2}$ and $-\nu-1 = -\frac{1}{2} + i\frac{\rho}{2}$) we need consider only $\rho_N > 0$.

Thus the reflectionless condition eliminates the supplementary series of representations $(0 < |s| < 1)$ and allows only the above denumerable set in the even principal series. This ties nicely with the fact that, of the two representations, only the even principal series is induced by a representation of a subgroup of $SL_2(\mathbb{R})$, in the present case the diagonal subgroup $\mathfrak{D}$, which describes time-translations. Thus time plays here the role of a parameter in the fiber and the stable states are the eigenstates under time-translation, that is to say, they are precisely our states $U^{\pm}$. This justifies the separation in the variables X^1 and X^4.

7.6 RESONANCE STATES

7.6.1 The points of the spectrum that, in general, contribute the most, locally, are those where $|\xi'(\Lambda)|$ has a local maximum. The corresponding eigenstates $U^{\pm}$ are therefore expected

to behave almost like excited states under external excitations and should exhibit resonance effects in scattering phenomena. We expect therefore that they should serve to describe the experimentally observed resonances.

We thus look for the points $\Lambda > 0$ where $|\xi'(\Lambda)|$ or $|\zeta'(\Lambda)|$ have a (local) maximum, with $\alpha - \frac{1}{2} = i\frac{\rho}{2}$ in (7.45), ρ real and non-zero.

Therefore let us consider

$$(7.52) \qquad \xi'(\Lambda) = \frac{1}{2}\,\mathrm{Im}\,\frac{\Gamma[\frac{1}{4}-i(\frac{\rho}{4}+\sqrt{\Lambda})]\Gamma[\frac{1}{4}+i(\frac{\rho}{4}-\sqrt{\Lambda})]}{\Gamma[\frac{3}{4}-i(\frac{\rho}{4}+\sqrt{\Lambda})]\Gamma[\frac{3}{4}+i(\frac{\rho}{4}-\sqrt{\Lambda})]},$$

$$(7.53) \qquad \zeta'(\Lambda) = -\frac{1}{2}\,\mathrm{Im}\,\frac{\Gamma[\frac{3}{4}-i(\frac{\rho}{4}+\sqrt{\Lambda})]\Gamma[\frac{3}{4}+i(\frac{\rho}{4}-\sqrt{\Lambda})]}{\Gamma[\frac{1}{4}-i(\frac{\rho}{4}+\sqrt{\Lambda})]\Gamma[\frac{1}{4}+i(\frac{\rho}{4}-\sqrt{\Lambda})]}.$$

Let $a \in \mathbb{R}$. We follow (p.45, [12]) and write in polar form

$$\Gamma(\tfrac{1}{4}k + \tfrac{1}{2}ia) = G_k(a)\, e^{i\gamma_k(a)}, \qquad k = 0,1,2,3,$$

where

$$\gamma_3 - \gamma_1 = \mathrm{tg}^{-1} \tanh \tfrac{1}{2}\pi a.$$

Using that $\overline{\Gamma(\frac{1}{4}k + \frac{1}{2}ia)} = \Gamma(\frac{1}{4}k - \frac{1}{2}ia)$ and setting $a = \frac{\rho}{2} + \sqrt{\Lambda}$, we get

$$(7.54) \qquad \xi'(\Lambda) = \frac{1}{2}\,\frac{G_1(2\sqrt{\Lambda}+\frac{\rho}{2})}{G_3(2\sqrt{\Lambda}+\frac{\rho}{2})} \cdot \frac{G_1(2\sqrt{\Lambda}-\frac{\rho}{2})}{G_3(2\sqrt{\Lambda}-\frac{\rho}{2})}\sin\beta,$$

$$(7.55) \qquad \zeta'(\Lambda) = \frac{1}{2}\,\frac{G_3(2\sqrt{\Lambda}+\frac{\rho}{2})}{G_1(2\sqrt{\Lambda}+\frac{\rho}{2})} \cdot \frac{G_3(2\sqrt{\Lambda}-\frac{\rho}{2})}{G_1(2\sqrt{\Lambda}-\frac{\rho}{2})}\sin\beta,$$

where

$$(7.56) \qquad \beta = \beta(\sqrt{\Lambda}) = \mathrm{tg}^{-1} \tanh \pi(\sqrt{\Lambda}+\tfrac{\rho}{4}) + \mathrm{tg}^{-1} \tanh \pi(\sqrt{\Lambda}-\tfrac{\rho}{4}).$$

Clearly $\beta = 0$ for $\Lambda = 0$ so that

(7.57) $$\xi'(0) = \zeta'(0) = 0.$$

As $\Lambda \to +\infty$, $\beta \to \frac{\pi}{2}$ and $\sin\beta \to 1$. On the other hand (by (5.46) in [12]) as $a \to \infty$

(7.58) $$\frac{G_3(a)}{G_1(a)} \sim \sqrt{\frac{a}{2}}\,\{1 - \frac{1}{(4a)^2} - \frac{19}{2(4a)^4} - \frac{631}{2(4a)^6} - \dots\}.$$

Hence as $\Lambda \to \infty$:

$$\zeta'(\Lambda) \sim \frac{1}{2}\,(\Lambda - \frac{\rho^2}{16})^{\frac{1}{2}} \sim \frac{\sqrt{\Lambda}}{2},$$

$$\xi'(\Lambda) \sim \frac{1}{2\sqrt{\Lambda}}.$$

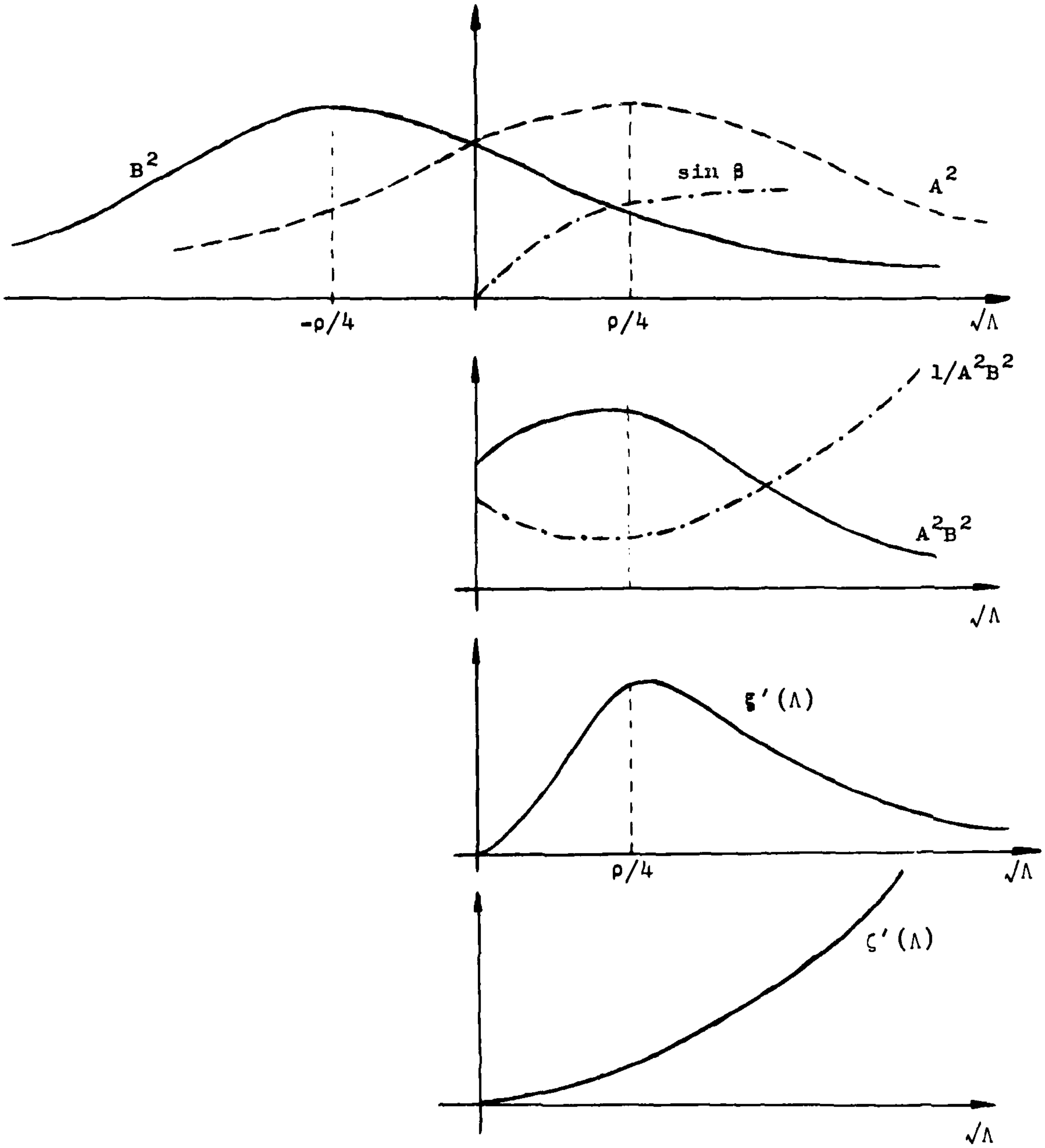
Therefore as $\xi'(0) = \xi'(+\infty) = 0$, it follows that $\xi'(\Lambda)$ has a point Λ_o of absolute maximum. From the known expression

$$\left|\frac{\Gamma(x+iy)}{\Gamma(x)}\right|^2 = \prod_{n=0}^{\infty}\{1 + \frac{y^2}{(x+n)^2}\} \qquad x \neq 0,-1,-2,\dots$$

we obtain, after a few simple calculations,

(7.59) $$\frac{G_1(a)}{G_3(a)} = \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}\left\{\prod_{n=0}^{\infty}\left[1 - \frac{(\frac{1}{2}+n)}{(\frac{3}{4}+n)^2}\cdot\frac{1}{1 + \frac{(\frac{1}{4}+n)^2}{(a/2)^2}}\right]\right\}^{\frac{1}{2}}$$

which shows that this quotient is (an even) decreasing function of a. On the other hand (7.56) shows that $\sin\beta(\sqrt{\Lambda})$ is a rapidly monotone increasing function, from the value 0 at $\sqrt{\Lambda} = 0$ to 1 at ∞. The following graphs indicate the behavior of the various factors in (7.54), where A and B are given by (7.60) and (7.61).



It is therefore clear that $\xi'(\Lambda)$ has a maximum near $\frac{\rho}{4}$ and that it has the shape shown in the figure above.

express the same phenomenon in terms of the usual (spherical) Legendre functions, more appropriate to spherical symmetry.

7.7.5 COMPLETING THE DESCRIPTION

We can now complete the description of the mechanism of resonance, on the basis of our results.

To that end let us recall from previous chapters, the precise meaning of a massive quanta, in our case the pion π^o. In (X^1,X^4)-space a standing pion consists of the statistical superposition of the isotropic waves $\varphi_{\pm}(X^1,X^4) = e^{im(X^1 \pm X^4)}$ with equal probabilities, giving rise to the average stationary states $e^{i(n+1/2)m'_oX^4} \cdot \sin(n+\frac{1}{2})m'_oX^1$. This is multiplied by $(X^2 \pm iX^3)^{1-s}$, thus originating the point-like higher order singularities of the associated phase function, and in accordance with the internal symmetry conditions of the geometry.

This then represents the pion as a point-like carrier of energy. One gets a corresponding spherical wave picture in concrete space on changing from X^1 into the r-coordinate (cf. (5.35), where ρ stands for X^1).

The moving single pion instead is simply the statistical superposition of the two waves in (X^1,X^4) space with constant probabilities say $\alpha > \beta > 0$, $\alpha+\beta = 1$. In this case the single pion will move with constant velocity in (X^1,X^4)-space, in the direction corresponding to the motion of the wave with the larger probability.

Let us follow the motion of the single pion in concrete space, looking however at its wave representation in the natural isotropic space (X^1,X^4).

7.7.4 COMMENTS

As long as the precise nuclear forces are not known, this analysis remains limited in scope. In fact, from our analysis we have obtained the resonances as solutions of the 2-dimensional wave equation in (r,t) space with a positive coupling k to (y,z)-space. We appealed to the reflectionless condition applied to the analytic continuation in r, of the (r,t) metric across $r \leq r_0$. Our resonance states are given by conical (Legendre) functions, no longer by Legendre functions with integer indices. Repetition of the same analysis for the full wave operator involving all four coordinates would bring in the axial symmetry effects, but this would be useless because in our case new phenomena arise (exchange forces, metric averaging, Higgs fields) in order for the metric to adjust itself to the boundary conditions at infinity, and have to be taken into account. Only after all this has been taken into full account can we think of repeating the above classical analysis. But at that stage we will have already collected most of the relevant information. Thus we see that our analysis represents a departure from the usual classical method, but nevertheless the analysis in (r,t)-space gives the known kind of results. In particular one should notice that the ℓ numbers of partial wave analysis become indeed imaginary ℓ numbers in our case cf. Section 7.11. Notice that the main point of departure is the fact that we characterized the resonances as eigenstates of the time-translation fiber subgroup $\mathfrak{A}$ in the even principal series representations of the symmetry group $SL_2(R)$, which bring in the conical Legendre functions, whereas the classical partial wave analysis tries to

$$(E-m-U)\varphi = \sigma p \chi$$
$$(E+m-U)\chi = \sigma p \varphi,$$

looking for bispinor solutions (in 4-vector form)

$$\psi = \binom{\varphi}{\chi} = \begin{pmatrix} f(r)\ \Omega_{j\ell m} \\ g(r)\ \Omega_{j\ell' m} \end{pmatrix}, \qquad \ell = j \pm \frac{1}{2}, \ \ell' = 2j-\ell$$

where

$$\Omega_{j\ell m} = \begin{pmatrix} \sqrt{\frac{j+m}{2j}}\ Y_{\ell, m-\frac{1}{2}} \\ \sqrt{\frac{j-m}{2j}}\ Y_{\ell, m+\frac{1}{2}} \end{pmatrix}$$

and a similar expression for $\Omega_{j\ell' m}$.

For $r \to \infty$ we have, for a central potential U decaying faster than $\frac{1}{r}$,

$$\psi \sim \sqrt{\frac{2}{\pi}}\frac{1}{r}\frac{1}{\sqrt{2E}} \begin{pmatrix} \sqrt{E+m}\ \Omega_{j\ell m} \sin(pr - \frac{\pi\ell}{2} + \delta \\ -\sqrt{E-m}\ \Omega_{j\ell' m} \sin(pr - \frac{\pi\ell}{2} + \delta \end{pmatrix}$$

Just like for the non-relativistic case, $e^{2i\delta} - 1$ determines the scattering amplitude and as a function of the energy has poles at the bound states of the particles (p.157 [2]). Under the assumptions on the potential U the behavior at ∞ is governed by the free equation in which case one gets that each component of the four vector ψ satisfies the equation

$$(\Delta + k^2)\psi = 0,$$

which coincides formally with the Schrödinger non-relativistic equation. Hence it is not surprising that the essential ingredients (phase shift, etc.) are basically the same in both cases.

are real then taking complex conjugate in k^2, in (7.80) and (7.81), it is easy to see that $S(\ell,k^2)$ defined by (7.92) satisfies

$$S(\ell,k^2)S(\ell,-k^2) = S(\ell,k^2)\cdot\overline{S(\ell,k^2)} = 1.$$

Thus if $S(\ell,k^2)$ has a simple pole at $k^2 = k_r^2 - i\frac{\Gamma}{2}$, k_r and Γ real, then we can factor it out as

$$S(\ell,k^2) = \frac{k^2-k_r^2-i\frac{\Gamma}{2}}{k^2-k_r^2+\frac{i\Gamma}{2}} S_o(\ell,k^2)$$

with $S_o(\ell,k^2)$ regular and non-zero near the zero and the pole. Thus the phase shift δ_ℓ near the pole is given by

$$\delta_\ell = tg^{-1}\left[\frac{\Gamma}{2(k_r^2-k^2)}\right] + \delta_o ,$$

δ_o slowly varying. This is the Breit-Wigner expression for the phase shift near a resonance. The non-real poles nearest to the real axis are called Regge-poles (see [9], [11], [3]).

7.7.3 THE RELATIVISTIC PICTURE

There are many possible ways to study this case. One is to derive the analyticity of $f(k^2,\Delta^2)$, where $\Delta^2 = 2k^2(1-\cos 2\theta)$ from the general axioms of quantum field theory (p.290 [11]).

The other would be to specify the field theory by a Hamiltonian and to compute the scattering amplitude through perturbation theory i.e. the analogue of the non-relativistic Born series (namely the Feynman method). More specifically one can consider the analysis directly on the time independent Dirac's equation (p.154 [2])

paring with (7.78) one gets for the scattering amplitude

$$(7.84) \qquad f(k^2,\cos\theta) = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{k} e^{i\delta_\ell} \sin\delta_\ell \, P_\ell(\cos\theta).$$

The coefficient

$$(7.85) \qquad a_\ell(k^2) = \frac{1}{k} e^{i\delta_\ell} \sin\delta_\ell$$

is called the partial wave amplitude.

Near $r = 0$, $V(r)$ and k^2 are dominated by $\frac{1}{r^2}$ so that the solution $y_\ell(r)$ must be asymptotic at 0 to the solution of

$$(7.86) \qquad \frac{d^2y_\ell(r)}{dr^2} - \frac{\ell(\ell+1)}{r^2} y_\ell(r) = 0.$$

The solutions are $y_\ell(r) = r^{\ell+1}$ and $y_\ell(r) = r^{-\ell}$ and only the first one satisfies the boundary condition $y_\ell(0) = 0$.

We now normalize the solutions of (7.80) at $r = 0$ by *requiring* that

$$(7.90) \qquad \frac{u_\ell(k^2,r)}{r^{\ell+1}} \to 1 \quad \text{as} \quad r \to 0.$$

At ∞ the normalized solution may be written, according to (7.83), in terms of outgoing and incoming waves, as

$$(7.91) \qquad u_\ell(k^2,r) \underset{r\to\infty}{\sim} \varphi^-(\ell,k^2)e^{ikr} + \varphi^+(\ell,k^2)e^{-ikr}.$$

The factors φ^- and φ^+ are simply the transmission and reflection coefficients.

Comparing (7.91) with (7.83) one gets

$$(7.92) \qquad e^{2i\delta_\ell} = (-1)^{\ell+1} \frac{\varphi^-(\ell,k^2)}{\varphi^+(\ell,k^2)} \equiv S(\ell,k^2).$$

The solutions $u_\ell(k^2,r)$, as well as φ^- and φ^+, are analytic functions of k^2 in appropriate domains. If $\ell(\ell+1)$ and V

where P_ℓ is the Legendre polynomial of degree ℓ, (7.77) reduces to the infinite set of ordinary differential equations

$$(7.80) \qquad \frac{d^2 y_\ell(r)}{dr^2} + [k^2 - V(r) - \frac{\ell(\ell+1)}{r^2}] y_\ell(r) = 0$$

with the boundary condition

$$(7.81) \qquad y_\ell(0) = 0.$$

Here we have set $\frac{2m}{\hbar^2} E = k^2$, $\frac{2m}{\hbar^2} U = V$.

The potential generally used is the Yukawa potential $V = \frac{e^{-\mu r}}{r}$, $\mu > 0$. Therefore for large r, the asymptotic behavior of $y_\ell(r)$ is given by disregarding V and $1/r^2$ with respect to k^2, when $k \neq 0$, so that (7.80) becomes

$$(7.82) \qquad \frac{d^2 y_\ell}{dr^2} + k^2 y_\ell(r) = 0, \qquad r \to \infty.$$

Its general solution is

$$y_\ell(r) \sim c_\ell \sin(kr + \alpha_\ell),$$

α_ℓ a constant phase.

On the other hand the solution of

$$\frac{d^2 y_\ell}{dr^2} + [k^2 - \frac{\ell(\ell+1)}{r^2}] y_\ell = 0$$

is given by Bessel functions, which behave as

$$y_\ell \sim c_\ell \sin(kr - \frac{\ell\pi}{2}), \qquad r \to \infty.$$

Thus if we set $\alpha_\ell = -\frac{\ell\pi}{2} + \delta_\ell$ we get

$$(7.83) \qquad y_\ell(r) \sim c_\ell \sin(kr - \frac{\ell\pi}{2} + \delta_\ell) \quad \text{as} \quad r \to \infty,$$

and δ_ℓ then expresses the phase shift caused by V in (7.80).

Using (7.83) in the expression (7.79) for ψ and com-

on their relative distance r and $\psi = \psi(r)$ is the wave-function describing their relative motion in the quantum-mechanical sense. This means that $|\psi|^2$ is supposed to describe the statistical average distribution of the incident particle over a very large number of identical two-particle interaction experiments with same initial conditions or, equivalently, the result of a single experiment with a very large number of mutually independent particles forming a beam of identical particles.

The solutions representing scattering must behave at ∞ like the sum of the incident plane wave $e^{i\vec{k}\cdot x}$ plus a scattered spherical wave $f(\theta,\varphi)\frac{e^{ikr}}{r}$, $r = |x|$, $k = |\vec{k}|$, where θ is the azimuthal angle with respect to the axis defined by $\vec{k}$, and φ is the corresponding latitude angle. The scattering amplitude $f(\theta,\varphi)$ describes the relative distribution of scattered particles on the surface of the spheres $r = \text{const}$. In fact due to the axial symmetry, $f = f(\theta)$ only.

Therefore the boundary condition at ∞ is

$$\psi(x) = e^{i\vec{k}\cdot x} + f(\theta)\,\frac{e^{ikr}}{r} + O\left(\frac{1}{r^2}\right), \qquad r \to \infty. \tag{7.78}$$

The ultimate goal is to determine $f(\theta)$, because this is what one measures experimentally. The problem (7.77), (7.78) is equivalent to the following integral equation

$$\psi(x) = e^{i\vec{k}\cdot x} - \frac{1}{4\pi}\int \frac{e^{ik|x-y|}}{|x-y|}\,V(y)\psi(y)\,dy.$$

In our case $\psi = \psi(r,\theta)$ only, so if we write the decomposition appropriate to spherical coordinates

$$\psi = \sum_{\ell=0}^{\infty} \frac{y_\ell(r)}{r}\,P_\ell(\cos\theta), \tag{7.79}$$

$$(7.74) \qquad T_{L_1} = \frac{c^2}{2m_p}[(m_\pi+m_p+m_\pi)^2 - (m_\pi+m_p)^2]$$
$$= \frac{c^2}{2m_p}[4m_\pi^2 + m_p^2 + 4m_\pi m_p - m_\pi^2 - m_p^2 - 2m_\pi m_p]$$
$$= [1 + \frac{3}{2}\frac{m_\pi}{m_p}]m_\pi c^2.$$

In general to get the proton excited to the n-th energy level i.e. $(p+n\pi)$ we need

$$(7.75) \qquad T_{L_n} = n[1 + (\frac{n}{2}+1)\frac{m_\pi}{m_p}]m_\pi c^2, \qquad n = 1,2,\dots .$$

As $m_\pi c^2 = 134.974$ MeV and $m_p c^2 = 938.256$ MeV, we have $\frac{m_\pi}{m_p} = 0.143856$ and so

$$(7.76) \qquad T_{L_1} = [1+0.2.1578]134.974 \text{ MeV} \cong 164.10 \text{ MeV}.$$

The rest mass of the resonance N^* so obtained is calculated by the corresponding total energy $c\sqrt{s}$ as given by (7.69) and (7.72), in the CM-system, which is the rest system of N^*.

At the lowest treshold kinetic energy (7.76) one thus obtains the first experimentally observed resonance $N^*_{3/2}(1238)$.

7.7.2 THE NONRELATIVISTIC WAVE PICTURE

The non-relativistic wave picture of the scattering process consists in studying the solution ψ of the time independent Schrödinger equation

$$(7.77) \qquad \Delta\psi + \frac{2m}{\hbar^2}[E-U]\psi = 0,$$

where $m = \frac{m_1 m_2}{m_1+m_2}$ is the reduced mass of the system incident particle-target, E is the energy level of excitation, $U = U(r)$ is the interaction potential between the two-particles depending

We now assume that m_2 is so large that the second particle may be assumed at rest with respect to the laboratory frame, i.e. we assume that the laboratory frame is defined by

$$\vec{p}^{(2)} = 0. \tag{7.71}$$

With respect to the laboratory frame we have, using (7.64), (7.65) and (7.71) in the defining expression for s

$$\begin{aligned} s &= \frac{1}{c^2}(E_1+E_2)^2 - (p^{(1)})^2 \\ &= \frac{E_1^2}{c^2} - (p^{(1)})^2 + \frac{E_2^2}{c^2} + 2\frac{E_2}{c^2}\cdot E_1 \\ &= m_1^2c^2 + m_2^2c^2 + 2m_2(m_1c^2 + T_1) \\ &= c^2(m_1+m_2)^2 + 2m_2\, T_1 , \end{aligned} \tag{7.72}$$

where T_1 is the kinetic energy of particle 1 in the laboratory frame.

Comparing with (7.70) one gets

$$T_1 = \frac{c^2[(m_3+m_4)^2 - (m_1+m_2)^2]}{2m_2}. \tag{7.73}$$

This is the laboratory treshold kinetic energy of particle 1 considered as impinging on the static particle 2 and gives the minimum kinetic energy necessary for the process to take place.

If we apply (7.73) to the process (7.63) where particle 1 is the pion, particle 2 is the proton, particle 3 is the one-level excited proton $(p+\pi)$, and particle 4 is the slowed-down pion, we get for the treshold kinetic energy of the pion:

These are the appropriate kinematical concepts in special relativity as shown by Einstein and Minkowski (p.39 [6]).

Consider therefore the collision of two particles with 4-momenta $p^{(1)}$, $p^{(2)}$ and rest masses m_1, m_2 respectively, which then become particles of 4-momenta $p^{(3)}$, $p^{(4)}$ and rest masses m_3, m_4 respectively.

Conservation of total 4-momentum implies

$$(7.66) \quad p^{(1)}+p^{(2)} = p^{(3)}+p^{(4)}$$

Let us concentrate on the Lorentz invariant $s := (p^{(1)}+p^{(2)})^2$. Consider the center of mass system (CM-system) defined by

$$(7.67) \qquad \vec{p}^{(1)} + \vec{p}^{(2)} = 0,$$

and consequently, from (7.65) also by

$$(7.68) \qquad \vec{p}^{(3)} + \vec{p}^{(4)} = 0.$$

In this referential, using (7.64), (7.66), (7.67), we get

$$(7.69) \qquad s = \frac{1}{c^2}(E_1+E_2)^2 = \frac{1}{c^2}(E_3+E_4)^2.$$

Let us compute the minimum amount of energy in order for the above process to occur: this corresponds to the newly formed particles being at rest in the CM-system *i.e.* to $\vec{p}^{(3)} = \vec{p}^{(4)} = 0$. In this case (7.64) gives $E_3 = m_3c^2$, $E_4 = m_4c^2$ so that (7.69) becomes

$$(7.70) \qquad s = c^2(m_3+m_4)^2.$$

A beam of pions impinge on target nucleons (say protons) and transfer to them their kinetic energy, exciting the protons i.e. taking them into a higher internal state, as defined in Chapter V. The occurrence requiring the minimum amount of energy corresponds to an originally unexcited proton that goes into its first excited state i.e. into a proton plus a new pion. In symbols:

$$\pi + p \rightarrow \pi + \underbrace{(p+\pi)}_{\text{excited proton}} = 2\pi + p. \tag{7.63}$$

7.7.1 THE RELATIVISTIC POINT PICTURE

As we shall see later on, far from the immediate vicinity of the particle, the average metric is essentially Minkowskian: $ds^2 = (dx^4)^2 - (dx^2)^2 - (dx^2)^2 - (dx^3)^2$. We can therefore apply the Minkowskian momentum and energy conservation laws to study the two-body (inelastic) scattering and apply the results to the above process. First let us shortly recall that if a point-particle has a 4-momentum P, then in $P = (\frac{E}{c}, p_1, p_2, p_3) \equiv (\frac{E}{c}, \vec{p})$, the first component times c describes its total energy and the three-vector $\vec{p}$ describes its usual 3-momentum, in the given referential. The Lorentz invariant P^2, divided by c^2, defines its <u>rest mass</u> m_o :

$$m_o^2c^2 := P^2 = \frac{E^2}{c^2} - p^2 , \tag{7.64}$$

where $p^2 \equiv (\vec{p})^2 = p_1^2 + p_2^2 + p_3^2$.

The kinetic energy T of the particle in the given referential, is defined by

$$m_oc^2 + T := E, \quad \text{i.e.} \tag{7.65}$$

$$T = \sqrt{m_o^2c^4 + c^2p^2} - m_oc^2. \tag{7.65'}$$

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Doing the calculations when $N = 2$, i.e. when $\rho_2 = \sqrt{23} \cong$ $\cong 4.79$, we obtain the values

a	$\sqrt{\Lambda}$	$\xi'(\Lambda)$
0.00	1.20	1.3553
.06	1.23	1.4588
.10	1.25	1.5016
.16	1.28	1.5419
.18	1.29	1.5467
.20	1.30	1.5473 ←
.22	1.31	1.5457
.26	1.33	1.5316
.30	1.35	1.5087
.36	1.38	1.4578
.40	1.40	1.4187

Therefore for $N = 2$, $a_{max} = a_2 = 0.20$ and $\sqrt{\Lambda}_{max} = \sqrt{\Lambda}_2 \cong 1.30$.

As the point Λ_{max} moves to the right with increasing ρ we thus conclude that

$$\frac{\rho}{4} + 0.10 \le \sqrt{\Lambda}_{max} < \frac{\rho}{4} + 0.11, \qquad N \ge 2.$$

We shall take the approximation

$$\sqrt{\Lambda}_{max} = \sqrt{\Lambda}_N = \frac{\rho_N}{4} + 0.10, \qquad N \ge 2.$$

Thus in general

$$\sqrt{\Lambda}_N = \frac{\rho_N}{4} + c_N, \qquad N = 1,2,\ldots, \tag{7.62}$$

$$c_1 \cong 0.09, \quad c_N \cong 0.10, \quad N \ge 2.$$

We now review some general facts about resonances.

7.7 MECHANISM OF RESONANCE

The resonance states obtained by pion-proton (or more generally pion-nucleon) scattering consist of the following.

$\cong \frac{\rho_1}{4} + 0.09.$

For $N \geq 2$ we shall now show that $\sqrt{\Lambda_N} \cong \frac{\rho_N}{4} + 0.10.$

Indeed $B^2 = \frac{G_1(a)}{G_3(a)}$ is decreasing for $a \geq 0$ in view of (7.61) and by (7.58) its derivative satisfies $(B^2)' \sim -\frac{1}{\sqrt{2a^3}}$, so it is slowly decreasing for larger a. When ρ increases the graph of $B^2(A^2)$ is displaced to the left (right, resp.). Therefore the larger ρ is the more the relative behavior of $\xi'(\Lambda)$ is governed by $H = \frac{G_1(2\sqrt{\Lambda} - \frac{\rho}{2})}{G_3(2\sqrt{\Lambda} - \frac{\rho}{2})} \sin \beta$. Now for $N = 2$ and $\sqrt{\Lambda} \geq 0$ we have $\rho_N \geq 4.79$ and $a \geq 2.39$ and thus $\sin \beta = 1.000$ already for $a = 2$ (see the table). Since $\sin \beta$ is increasing it follows that it is practically one for $N \geq 2$. Thus for $N \geq 2$, H is essentially a function of $a = 2\sqrt{\Lambda} - \frac{\rho}{2}$ alone. Therefore we can find the point of maximum of H once and for all. Tabulating H near the previously obtained value $a_1 = 0.18$ we obtain

a	H
0.00	2.0921
.06	2.2659
.10	2.3436
.16	2.4211
.18	2.4336
.20	2.4398
.22	2.4403 ←
.24	2.4366
.26	2.4280
.30	2.3989
.36	2.3337
.40	2.2802

The maximum of H is at $a = 0.22$. Thus certainly $\sqrt{\Lambda}_{max} < \frac{\rho}{4} + 0.11$ for $N \geq 2$.

7.6.2 CALCULATION OF THE PEAK FREQUENCY

Let us denote by Λ_N the above peak frequency when $\rho = \rho_N$ is given by (7.51).

We write

$$(7.60) \qquad A = \left| \frac{\Gamma(\frac{1}{4}+i\frac{a}{2})}{\Gamma(\frac{3}{4}+i\frac{a}{2})} \right|^{\frac{1}{2}} = \left| \frac{G_1(2\sqrt{\Lambda}-\frac{\rho}{2})}{G_2(2\sqrt{\Lambda}-\frac{\rho}{2})} \right|^{\frac{1}{2}}$$

and

$$(7.61) \qquad B = \left| \frac{\Gamma(\frac{1}{4}+i\frac{a+\rho}{2})}{\Gamma(\frac{3}{4}+i\frac{a+\rho}{2})} \right|^{\frac{1}{2}} = \left| \frac{G_1(2\sqrt{\Lambda}+\frac{\rho}{2})}{G_3(2\sqrt{\Lambda}+\frac{\rho}{2})} \right|^{\frac{1}{2}} ,$$

so that $\xi' = A^2B^2 \cdot \sin\beta$ and $\zeta' = A^{-2}B^{-2} \cdot \sin\beta$. We then make use of Table V on p.226 [12]. Taking in particular $N = 1$ we have $\rho_1 = \sqrt{7} \cong 2.646$ and $\sqrt{\Lambda} = \frac{a}{2} + 0.661$. We thus get the following table of values

a	$\sqrt{\Lambda}$	sin β	ξ′
0.00	0.66	0.7069	1.8355
0.06	0.69	0.7707	1.9649
0.10	0.71	0.8063	2.0169
0.16	0.74	0.8554	2.0605
0.18	0.75	0.8695	2.0636←
0.20	0.76	0.8822	2.0613
0.22	0.77	0.8938	2.0543
0.26	0.79	0.9145	2.0287
0.30	0.81	0.9316	1.9908
0.36	0.84	0.9516	1.9163
0.40	0.86	0.9617	1.8599
0.60	0.96	0.988	1.563
0.80	1.06	0.997	1.312
1.00	1.16	0.999	1.123
2.00	1.66	1.000	0.671

It is clear that the peak value occurs at $\sqrt{\Lambda_1} \cong 0.75 \cong$

Suppose the larger probability α corresponds to the incoming spherical wave in concrete space (r,t). The pion point-like singularity will then approach the shell of the particle with total energy $E_1 = m_\pi c^2 + T_L$, where T_L is its kinetic energy. When it reaches the shell the following phenomena occur:

i) the wave with probability amplitude α splits into two similar waves, one with amplitude $\gamma = \alpha-\beta$, and the other with amplitude β. The last one forms with the remaining outgoing wave, which also has amplitude β, a single pion standing at the shell of the particle;

ii) the remaining γ wave, which is the carrier of the excess energy T_L, disappears giving rise instead to a standing stationary state of the proton $e^{i(n'+1/2)m_o'X^4} \cdot \sin(n'+\frac{1}{2})m_o'X^1$, inside the particles plus a standing stationary non-harmonic state solution of the wave equation outside the particle, corresponding to one of the resonances we have been studying.

There are three observations to be made. The first one is that the vanishing of the incoming γ photon wave and the simultaneous sudden appearance of the two stationary states, one inside and the other outside the particle, as well as the entering of the whole system into a sudden stationary state, is in line with the stochastic description brought out by L.C. Young's theorem on generalized curves as the correct descriptors for the solutions of evolution equations of Lagrangian processes, and corresponds to the following sudden changes of state:

a) the $\gamma(t) = \gamma$ $(t < t_o)$ probability jumping at $t = t_o$ into $\gamma(t) = 0$ $(t > t_o)$;

b) the internal state of the proton jumping at $t = t_o$ from unexcited to the above mentioned internal state of energy level n';

c) the external field of the particle, in addition to the existing field corresponding to the standing pion, jumping from non-excited into a resonance state.

The second observation has to do with the reason why the resonant state should be considered as an external state of the particle: the point is that the condition (5.63) of zero radial derivative at $r = 0$ that is satisfied by the harmonic internal states has not been imposed as a boundary condition for the resonant states. This condition is absolutely necessary in order to have stable stationary states inside the shell and can be dispensed with only for fields <u>outside</u> the shell. Therefore the resonant state must set in outside the particle.

Thirdly, as we shall see in Chapter X, even if the pion has been emitted by another particle, near the target its wave

behaves as an incoming wave of the target particle, which means we can consider the impinging pion as eventually described by the target particle geometry.

The so called resonance constitutes therefore the stationary state formed by the whole system of excited proton plus the slowed-down incoming pion and plus the resonant state field outside the shell of the particle.

7.8 MASS OF THE RESONANCES

7.8.1 The energy associated with the states (7.40), or with their superpositions that behave as the standing states (7.40'), is given by

$$E_r = 2c\,\hbar\,\mu\,\sqrt{\Lambda}.$$

Using the expression (5.67) for $\mu = \sqrt{2\lambda}$ this becomes

$$E_r = \frac{\sqrt{\Lambda}}{\pi} \cdot E'_o \cdot \log \coth R/2, \tag{7.93}$$

where $E'_o = m'_o c^2$ is the mass of the massive quanta in energy units. This is the energy associated with the field (7.40) outside the shell. This field is superposed to the field of the captured massive quanta which has energy E'_o and is of the form (5.69). The energy squared E^2_{out} of the total field outside the particle is thus $E^2_{out} = E'^2_o + E^2_r$.

The internal energy consists of the energy $E_{n'}$ of the n'-th excited state (5.69), plus the energy $E_o = m_o c^2$ corresponding to the mass of the geometry, i.e.

$$E_{in} = n' E'_o + E_o .$$

The total mass E_Λ of the resonance is thus $E_{out} + E_{in}$ and therefore we get, using (6.47),

$$E_\Lambda = [n' + \sqrt{1 + (\frac{2n_o+1}{1+|s_o|})^2}\ \Lambda]E_o' + E_o . \qquad (7.94)$$

Here n_o, $|s_o| + 1$ are the integers characterizing the *geometry*, according to (6.47).

Expressing further E_o'/E_o according to (6.0) we obtain (7.0), with $\Lambda = \Lambda_N$ in it.

In particular if we equate this energy with the initial energy of the system, namely $E_o' + T_L + E_o$, where T_L is the kinetic energy of the incident pion, we get

$$T_L/E_o' = (n'-1) + \sqrt{1 + (\frac{2n_o+1}{|s_o|+1})^2}\ \Lambda_N . \qquad (7.95)$$

7.8.2 STABILITY OF THE EXCITATION LEVELS

In the case of the proton the internal energy state with higher stability is the $n=3$ energy level corresponding to the energy of three pions π^o, *for the group-theoretical reasons* given in Chapter VI. We must therefore expect that, as a rule, a system of 3 pions π^o is in general the most stable configuration of pions even if the system is *not* precisely the E_3 - internally excited state.

In the case of resonances we have a system formed by the slowed-down pion at the shell of the particle plus the n' pions corresponding to the internal energy level $E_{n'}$. Therefore this configuration should be <u>more stable precisely when</u> $n' = 2$.

If we consult a table of decay modes of the resonances we see that, indeed, with exception of N(1238), wich decays

into p+π (100%), most of the remaining resonances present a partial decay into p+2π. There are no decays into more than two pions (the third pion being converted back into kinetic energy to set these two pions in motion).
According to (7.75) and (7.76) the laboratoy treshold kinetic energy necessary to get the $n' = 1$ and $n' = 2$ internal energy levels are, respectively,

$$T_{L_1} \cong 1.216\ E'_o \qquad (= 164.10\ \text{MeV}),$$
$$T_{L_2} \cong 2.575\ E'_o \qquad (= 347.62\ \text{MeV}),$$

where E'_o is the mass of the pion in energy units.

On the other hand the kinetic energy, in multiples of E'_o, available from the total energy of the resonance, is, according to (7.95),

	N=1	N=2	N=3	N=4	N=5
n'=1	1.196	1.514	1.876	2.265	2.668
n'=2	2.196	2.514	2.876	3.265	3.668

and so on.

We see that $2.196 > T_{L_1}/E'_o = 1.216 > 1.196$, so that, as long as the incident pion has energy above $1.196\ E'_o$ and below $2.196\ E'_o$, the resonance created must correspond to $n'=1$ (first internal energy level) and $N = 1$ (first resonant state). Even though the $n' = 1$ energy level is less stable than the $n' = 2$ level, the energy provided is just enough for the first resonance N(1238) to occur.

However for $N = 2$,

(7.96) $$T_{L_2}/E'_o = 2.575 > 2.514,$$

so that, even if we lower the kinetic energy of the incident pion to its treshold value T_{L_2}, we can form the $(N=2,\ n'=2)$-resonance. Therefore for $N \geq 2$ the internal energy state $n' = 2$ will always take place preferentially (provided the kinetic energy available is enough in each case) as the $n' = 1$ case is less stable.

On the other hand for the kinetic energy below the 2.196 threshold we see from the above table that $N \leq 3$ may occur with $n' = 1$, as far as the energy balance is concerned. In fact however for $n' = 1$ only the $N = 1$ case seems to occur experimentally. This can be explained by the argument that although presumably less stable than the case $n' = 2$, it is the resonance with *minimum* total energy and so it prevails below the 2.196 threshold.

7.8.3 MASS FORMULAE

According to (7.51), (7.62) and (7.94) the resonance masses may be computed by

(7.97) $$E_N/E_o = 1 + [n' + \sqrt{1 + (\frac{2n_o+1}{1+|s_o|})^2}\ \Lambda_N]E'_o/E_o, \quad N = 1,2,\ldots$$

(7.98) $$\Lambda_N = \sqrt{4N(N+1)-1} + c_N, \quad c_1 \cong 0.09, \quad c_N \cong 0.10 \quad N \geq 2.$$

Here n_o, $1+|s_o|$ are the integers characterizing the given geometry and $n' \leq \max(0, n_o-1)$, preferably $n' = n_o-1$.

If we replace E'_o/E_o by its expression (6.0) we obtain (7.0).

7.9 BARYON RESONANCES

Setting $n_o = 3$, $1+|s_o| = 8$ in (7.97), or in (7.0), we obtain the following table for the baryon resonances

N	$\sqrt{\Lambda}$	E_N(MeV)	Experimental values: Omnes	Experimental values: Frazer
1	0.75	$(n'=1)$ 1235	1236	1236
2	1.30	$(n'=1)$ 1273 $(n'=2)$ 1413	unobserved	
3	1.81	$(n'=2)$ 1461	1460	
4	2.32	" 1514	1515	1518 ± 10
5	2.83	1568	1525	
6	3.33	1624	1630	
7	3.83	1681	1675	1688
8	4.34	1738	1715	
9	4.84	1795	1785	
10	5.34	1853	1880	
11	5.84	1911	1905	1924
12	6.34	1969	1940	
13	6.84	2027	-	
14	7.34	2086	-	
15		2144	-	
16		2203	-	2190
17		2261	-	
18		2320	-	
19		2379	-	2360
20		2437	2420	
21		2496	-	
22		2555	-	
23		2614	-	
24		2672	2650	2645 ± 10
25		2731	-	
26		2790	-	
27		2849	2850	2825 ± 15
28		2908	-	
29		2967	-	
30		3026	3030	
31		3085	-	
32		3143	-	
33		3202	3230	
34		3261	-	

Here we have taken into account the considerations in 7.8.2, regarding the value of n'. In particular for $N \geq 3$ we only considered $n' = 2$.

We see that the agreement between the theoretically computed values and the experimental ones is very good at the low energies and that, even at higher energies, there are some good results. As remarked in Frazer p. 51 the experimental values are subject to variations due to the way that the mass is computed from the experimental data and also, at higher energies, due to the background noise caused by the presence of the lower energy resonances.

There are many undetected resonances at higher energies but the really noteworthy absence is that corresponding to $N = 2$, which has low energy. It is clear that the $n = 1$ state is always occupied at the lower energy $N = 1$, as mentioned before. The $(N=2, n=2)$ case with mass 1413 may be absent due to the experimental set up: From 7.8.2 we see that it requires the threshold kinetic energy $2.514\ E_o'$ which is smaller than but very close to $T_{L_2} = 2.575\ E_o'$. Under experimental conditions it is clear that one has to use values somewhat larger than T_{L_2}, and may be they reach the relatively close level $2.876\ E_o'$ corresponding to $N = 3$, which would then explain the appearance of the N(1461) resonance and the absence of the N(1413) one. However there is no doubt that, with the presently available theoretical values of the resonance levels, one should be able to set up experiments to detect their existence, even if they turn out to be less stable in the final set up.

For the baryons we may write, from (7.97) and (7.98), the following approximations $(N > 1)$:

$$\sqrt{\Lambda_N} = \frac{N}{2} + 0.35 - \frac{0.125}{N} + \frac{0.031}{N^2} + 0(\frac{1}{N^3}) \tag{7.99}$$

$$E_N = 1249.55 + 59.051\ N + \frac{1.395}{N} - \frac{1.120}{N^2} + 0(\frac{1}{N^3}) \quad \text{(MeV)}. \tag{7.100}$$

It is clear that for $N \geq 2$ $\sqrt{\Lambda_N}$ and E_N are practically linear functions of N.

7.10 ELECTRON RESONANCES

In previous sections we have described resonances of the proton on the basis of the laboratory experiments of scattering of pions by target protons and of the peak frequencies in the spectral density of the associated Schrödinger operator.

For several reasons no corresponding experimental set up can be repeated with the same ease for the electron. Among them:

i) the instability of the $n > 0$ levels of excitation for the electron which forbids it from interacting strongly, so that it has no massive quanta formed by two photons, like the π^o for the proton;

ii) even if, under special circumstances, one could operate with such photonic massive quanta, there is still the fact that its mass $m'_o = 197.2\ m_e$ is much bigger than that of the electron (while the pion is about 6 times lighter than the proton) so that scattering would more likely occur the other way around.

Nevertheless one should notice that the muon decays into

the electron plus a pair of neutrinos

$$\mu^- \rightarrow e^- + \nu\bar{\nu}$$

and a pair of neutrinos with opposite helicities describes a massive quanta just as two photons do, cf. Chapter III.

Although such a pair $\nu\bar{\nu}$ does not belong in the discrete series representation (cf. Section 6.10) there remains the possibility that it fulfills a stability condition corresponding to some other kind of symmetry, for instance the $SU_2 \times U_1$ symmetry of electroweak interactions (Chapter X).

Because $n_o = 0$ for the electron this kind of difficulty will always arise when considering the states of the electron for we have to work with objects that are well defined but found to be unstable. As in Chapter V we will have to assume that the massive quanta $\nu\bar{\nu}$ replaces the photonic massive quanta E_1 of the electron, and we will have to find later on what makes it stable. At any rate considerations of spin, i.e. of the symmetry properties under rotation in the spatial sections of the Minkowski background geometry will have to enter in our subsequent analysis, at some stage. Up to the present this has not entered in our analysis for we are still considering the symmetries of the nuclear field alone.

It should be emphasized that to have a complete picture there will have to be a conciliation between the requirements of the symmetries of the particle geometry and those of the Minkowski background. At present we are considering only the first ones, although in part, like in Section 7.7.1, we have already appealed to considerations on the background.

The fact that we have atributed to the massive quanta $\nu\bar{\nu}$ the same mass as that of the E_1 state of the electron and obtained coherent results (in particular the confirmation that $n = 0$ is the most stable state) seems to indicate that the neutrinos forming the above massive quanta are described by the additive superposition of the <u>same</u> states that enter multiplicatively in E_1 in (6.64), i.e. by the states

$$\varepsilon(y+iz)^{1-s} + e^{i\mu(s-1)(X_4 \pm X_1)},$$

which obviously are still harmonic solutions of $\Delta_2\varphi = 0$ although they do not belong in the discrete series (i.e. are not eigenstates of K).

Under these circumstances we now consider resonance states for the electron, formed just like those for the proton, with the sole difference that always $n' = 0$ for the electron and that instead of the photonic massive quanta E_1 standing at the shell of the electron there will be there a neutrino pair $\nu\bar{\nu}$ with the same mass m_o'.

Since the functions $\xi'(\Lambda)$ and $\zeta'(\Lambda)$ and the peak abcissae $\sqrt{\Lambda_N}$, $N = 1,2,\ldots$ do not depend on the particle's nature, the formulae for the electron resonances is simply (7.94) with $n' = 0$, $n_o = 0$ and $1 + |s_o| = 3$. This yields

$$(7.101) \qquad E_N = [1 + 197.205\sqrt{1+\Lambda_N/9}]E_e, \qquad N = 1,2,\ldots$$

where E_e is the electron mass, and Λ_N is given by (7.98).

Computing (7.101) we obtain the following values

N	$\sqrt{\Lambda_N}$	E_N(MeV)	
1	0.75	104.386	← $\mu^\pm$
2	1.30	110.339	
3	1.81	118.205	
4	2.32	127.900	
5	2.83	138.535	← $\pi^\pm$
6	3.33	151.068	
7	3.83	163.933	
8	4.34	177.735	
9	4.84	191.790	
10	5.34	206.256	
14	7.34	266.868	
20	10.34	362.291	
27	13.75	473.088	
28	14.35	492.818	← $K^\pm$
29	14.85	509.274	
30	15.35	525.753	
31	15.85	542.250	
40	20.35	691.37	
44	22.35	757.91	
45	22.85	774.56	
50	25.35	857.90	
52	26.35	891.27	← $K^{*\pm}$
55	27.85	941.36	
56	28.35	958.06	← η'

The values

$$E_1 = 104.386$$

$$E_5 = 138.535$$

$$E_{28} = 492.818$$

$$E_{52} = 891.27$$

$$E_{56} = 958.06$$

compare well with the masses (Table 1.2 [8])

$$E_{\mu^\pm} = 105.659$$

$$E_{\pi^\pm} = 139.580$$

$$E_{K^\pm} = 493.78$$

$$E_{K^{*\pm}} = 891 \pm 1$$

$$E_{\eta'} = 959 \pm 2,$$

all in MeV.

We thus arrive at the conclusion that μ^-, π^-, K^- and K^{*-} are resonances of the electron, corresponding to $N = 1,5,28,52$ respectively, according to the model that we have described earlier. This model is in accordance with the decay modes

$$\mu^- \to e\,\nu\bar{\nu} \qquad (100\%)$$

$$\pi^- \to \mu^-\,\nu \qquad (\sim 100\%)$$

$$K^- \to \mu^-\,\nu \qquad (63.1\%)$$

$$\pi^-\pi^0 \qquad (21.5\%)\ ,$$

$$K^{*-} \to K^-\pi^0 \qquad (\sim 100\%)$$

as they all decay ultimately into an electron.

The neutrino ν in the decay of π^- and K^- is the carrier of the difference energy of the resonance state formed outside of the shell of the geometry, whereas the massive quanta $\nu\bar{\nu}$ carries the mass m_o' plus the kinetic energy corresponding to the mass of the first resonance state $N = 1$. Thus this explains why the mass of the muon is not m_e+m_o' as it would if it were an excited state of the electron.

As remarked before we still need to study the symmetry requirements of the background geometry as well as the unitary symmetries associated with the various representations of the symmetries of the internal geometry, to complete the analysis. This will be done later on.

To close we notice an interesting fact. The π^- appears here as a state of the electron. Since $\pi^-\pi^+ = 2\pi^o$ and since π^o is a massive quanta of the proton geometry we face here transmutation of particles, considered as elementary geometries, into massive quanta, which in first instance are objects of a distinct nature. However as described in 4.3.7, in terms of their twistor description they are objects of the same basic nature, and differ only by the nature of the singularities of their respective twistor fields. It just suffices that singularities of one kind develop into those of the other in order to have a transformation of particles (as geometries) into massive quanta and vice-versa.

It thus seems that we may have here a first clue to the connection between the structures of the proton and of the electron, since the basic massive quanta of one turns out to be a resonance of the other.

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7.11 COMPARISON WITH THE PARTIAL WAVE ANALYSIS

The operator equation in (7.10) is given, in the concrete (r,t) coordinates by

$$e^{\mu|r-r_o|} \frac{d}{dr} (e^{\mu|r-r_o|} - e^{-\mu|r-r_o|}) \frac{d\varphi}{dr} - \frac{p\varphi}{1-e^{-2\mu|r-r_o|}} = k\varphi$$

i.e.

$$\frac{d^2\varphi}{dr^2} + \mu\coth\mu|r-r_o| \cdot \frac{d\varphi}{dr} - \frac{p\varphi}{4\sinh^2\mu|r-r_o|} = \frac{k\varphi}{e^{2\mu|r-r_o|} - 1} \tag{7.102}$$

Following the same procedure as in (p.128 [9]) we remove the first derivative $\frac{d\varphi}{dr}$ by introducing the new function y through $\varphi = \frac{y}{(\sinh\mu|r-r_o|)^{1/2}}$, in (7.102), which yields

$$\frac{d^2y}{dr^2} + \{-\frac{\mu^2}{4} + \frac{\mu^2(1+4\Lambda)}{4\sinh^2\mu|r-r_o|} - \frac{k}{2}\frac{e^{-\mu|r-r_o|}}{\sinh\mu|r-r_o|}\}y = 0. \tag{7.103}$$

This is of the form (7.80) with the modified Yukawa potential $V = \frac{k}{2}\frac{e^{-\mu|r-r_o|}}{\sinh\mu|r-r_o|}$ and the centripetal (instead of centrifugal) potential $\frac{\mu^2(1+4\Lambda)}{4\sinh^2\mu|r-r_o|}$.

Besides this we have a constant term $-\frac{\mu^2}{4}$ corresponding to a negative energy.

As $r \to \infty$ the solution of (7.103) behaves as the solution of $\frac{d^2y}{dr^2} = \frac{\mu^2}{4} y$, which is of the form $Ae^{\frac{\mu}{2}r} + Be^{-\frac{\mu}{2}r}$ and not as (7.83).

Near the shell the centripetal term $\sinh^2\mu|r-r_o| \sim \sim \mu^2|r-r_o|^2$ predominates over the Yukawa potential and over the constant term so that y behaves there as the solution of

$$(7.104)\qquad \frac{d^2y}{dr^2} + \frac{\Lambda+1/4}{(r-r_o)^2}\, y = 0,$$

whose linearly independent solutions are

$$(7.105)\qquad y_\pm = |r-r_o|^{\frac{1}{2} \pm i\sqrt{\Lambda}} .$$

Consequently near r_o φ behaves as

$$(7.106)\qquad \varphi_\pm \sim c|r-r_o|^{\pm i\sqrt{\Lambda}} = c\ e^{\pm i\sqrt{\Lambda}\,\log|r-r_o|} .$$

From (5.37), for $r \sim r_o$, we have

$$X^1 = \rho \sim -\frac{1}{2\mu}\log|r-r_o| + \text{const.}$$

so that

$$\varphi_\pm \sim \text{const}\ e^{\mp 2\mu\sqrt{\Lambda}\, i X^1}$$

which agrees with 3. in 7.2.3, as it should.

The present solution (7.106) is completely different in nature from $|r-r_o|^{\ell+1}$ which is the partial wave solution near r_o, according to (7.90).

We thus see that the usual partial wave analysis as presented in 7.7.2 fails to apply to our present situation.

Notice that the $|r-r_o|^{\ell+1}$ behavior at $r = r_o$ becomes in our case $|r-r_o|^{i\sqrt{\Lambda}}$: while $\ell+1$ runs over the integers, $\sqrt{\Lambda}$ has the values $\frac{N}{2} + 0.35 - \frac{0.125}{N} + O(\frac{1}{N^2})$ for $N \geq 2$ and so practically runs over the purely imaginary half-integers.

That the ℓ exponents should eventually run over complex numbers when dealing with resonances has been suspected long ago in connection with Regge poles.

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This report is the second part of a general theory purporting to describe the mathematical structure of the elementary particles, deriving it from first principles. It consists of Chapters VI and VII, continuing the MRC Technical Summary Report #2581, October 1983. In them we study the implications of the $SL_2(\mathbb{R})$ transformation group of the particles geometry. In Chapter VI we show

(cont.)

ABSTRACT (cont.)

how the discrete series of representations implies the quantization of the geometries and in particular why the electron does not interact strongly. In Chapter VII we obtain the resonances as states corresponding to the principal series of representations.

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